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ERDŐS PROBLEM 1062(II): IRRATIONALITY OF A DIVISIBILITY DENSITY

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ABSTRACT. Let $f(n)$ be the largest cardinality of a subset of $\{1,\ldots,n\}$ in which no member divides two distinct other members. The accepted Lean submission proves that $f(n)/n$ converges to an irrational real number, answering part (ii) of Erdős Problem 1062. We describe its exact finite counting formula, derive the density limit from a summable series of rank increments, and present an explicit coefficient expansion. Assuming the density rational produces unusually small, nonzero linear forms in rational powers of two and three; a three-place covering theorem proved in the source rules them out. We give the complete final analytic deduction and identify the longer formal inputs precisely. The paper is an exposition of the accepted artifact, not a replacement for its full formal development.

1. The question and the accepted result

Call a set $A \subseteq \mathbb{N}_{>0}$ *fork-free* if there do not exist three distinct elements $a, b, c \in A$ with $a \mid b$ and $a \mid c$. Equivalently, every member has at most one distinct proper multiple inside A. Define

$$f(n) = \max\{|A| : A \subseteq \{1,\ldots,n\},\ A \text{ is fork-free}\}.$$

The maximum exists because the interval is finite and the empty set is admissible. A pair $\{a, 2a\}$ is allowed; a triple $\{a, 2a, 3a\}$ is forbidden. The direction of divisibility matters: the condition concerns one divisor and two multiples, not one common multiple of two divisors.

Theorem 1.1 (Accepted target)**.** *There exists an irrational $L \in \mathbb{R}$ such that*

$$\lim_{n\to\infty} \frac{f(n)}{n} = L.$$

Limits are taken through positive integers. The interval $\{\lfloor n/3\rfloor + 1,\ldots,n\}$ is fork-free: two distinct proper multiples of a member a would include one at least $3a > n$. Consequently $\lceil 2n/3\rceil \le f(n) \le n$.

1.1. **Historical scope.** Guy's second edition (1994), section B24, page 80, explicitly asks whether $\lim f(n)/n$ is irrational [1]. This supports the announcement wording "open since at least 1994": a documented 32-year history in 2026, without asserting the first-proposal date. Lebensold's earlier bounds concern the same extremal problem [2]; their publication date alone does not date the irrationality question. Davis's April 2026 paper proves existence and effective computability of the density, while expressly leaving its irrationality open [3].

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Key words and phrases. Divisibility, extremal sets, irrationality, density, Lean.

Submitted by JenW1N. Working proof exposition and source guide prepared with Codex assistance from the accepted Lean submission. The full combinatorial construction and arithmetic covering theorem are proved in the linked formal source and summarized here. This prose is not itself kernel-verified.

2. The finite combinatorial formula

Every positive integer has a unique expression

$$m = q2^a3^b, \qquad a, b \geq 0, \quad \gcd(q, 6) = 1.$$

For a fixed q, divisibility among the corresponding integers is coordinatewise comparison of the lattice pairs (a, b). This gives useful local bounds, but the components are *not* independent: divisibility may also run between different cores. The submitted construction accounts explicitly for this obstruction.

For $t \geq 1$, set $\ell_b(t) = \lfloor \log_b t \rfloor$ and define

$$R(t) = \ell_3(t) + 1 + \left\lfloor \frac{\ell_2(t)}{2} \right\rfloor + \eta(t), \tag{1}$$

$$\eta(t) = \mathbf{1}_{\{\ell_2(t) \text{ odd},\ 2 \cdot 3^{\ell_3(t)} \leq t\}}. \tag{2}$$

Set $R(0) = 0$. The source proves that a fork-free subset of $\{2^a3^b : 2^a3^b \leq t\}$ has at most $R(t)$ elements. The additional correction is the indicator

$$B(t) = \mathbf{1}_{\{\exists j, b \geq 0:\ 4^j \leq t,\ 4t < 5 \cdot 4^j,\ 10 \cdot 3^b \leq t < 12 \cdot 3^b\}}. \tag{3}$$

Write $g(t) = R(t) - B(t)$, so $g(0) = 0$.

Proposition 2.1 (Finite identity supplied by the formal proof)**.** *For every $n \geq 0$,*

$$f(n) = \sum_{\substack{1 \leq q \leq n \\ (q,6)=1}} g\left(\left\lfloor \frac{n}{q} \right\rfloor \right). \tag{4}$$

The exact formula combines `f_eq_globalUpper` with `globalUpper_cast_sum`.

2.1. The upper bound and the factor-five loss.

The rank bound (1) applies to the part of an admissible set in each core. If the cutoff for q lies in the window (3), the source compares the components q and $5q$. Simultaneous saturation would force a member together with its double and its fivefold multiple, violating the fork-free condition. There is therefore a combined loss of at least one.

Let $Q(n) = \{1 \leq q \leq n : (q, 6) = 1\}$ and $E(n) = \{q \in Q(n) : B(\lfloor n/q \rfloor) = 1\}$. The source proves that $q \mapsto 5q$ is injective on $E(n)$, that its image lies in $Q(n)$, and that the image is disjoint from $E(n)$. Thus these loss pairs do not overlap. Summing gives

$$|A| + |E(n)| \leq \sum_{q \in Q(n)} R(\lfloor n/q \rfloor).$$

Maximizing over A proves the upper bound in (4). The disjointness is essential: without it the same missing element could be counted more than once.

2.2. Attainment and cutoff certificates.

The reverse inequality is the substantial constructive part of the formal development. It uses parity-dependent windows of two-three smooth numbers. Within a window the only possible proper multiple of a selected element is prescribed by its parity class; this proves the local fork-free property. Cutoff certificates coordinate these choices across the cores and attain the corrected global count.

The source proves the availability of the certificates by a finite base argument and strong induction. Failure above the base threshold would create one of two explicitly defined closure configurations; both configurations are ruled out in the submitted file. The declarations `closureRaw_false`, `cutoffAvailability`, and `f_eq_globalUpper` complete the construction.

3. A convergent series for the density

Define the discrete increments

$$\Delta_j = g(j+1) - g(j), \qquad j \geq 0,$$

and the core-counting function $C(m) = |\{1 \leq q \leq m : (q,6) = 1\}|$. Inclusion–exclusion gives the elementary identity

$$C(m) = m - \lfloor m/2 \rfloor - \lfloor m/3 \rfloor + \lfloor m/6 \rfloor = m/3 + O(1).$$

Lemma 3.1 (Finite increment expansion)**.** *For every $n \geq 1$,*

$$f(n) = \sum_{j=0}^{n-1} \Delta_j C\left(\left\lfloor \frac{n}{j+1} \right\rfloor\right).$$

Proof. Since $g(0) = 0$, telescoping gives $g(m) = \sum_{j<m} \Delta_j$. Insert this into (4) and interchange the two finite sums. The increment Δ_j occurs exactly for cores satisfying $j + 1 \leq \lfloor n/q \rfloor$, equivalently $q \leq \lfloor n/(j+1) \rfloor$. Their number is the stated C. ☐

The formal development establishes the absolute weighted summability

$$\sum_{j \geq 0} \frac{|\Delta_j|}{j+1} < \infty. \tag{5}$$

It separates increments of R from those of the correction B. For the correction, changes can occur only at endpoints of four geometric families: 4^j, $5 \cdot 4^j$, $10 \cdot 3^b$, and $12 \cdot 3^b$. Their reciprocal sums converge. The corresponding rank-increment analysis is also supplied in the source.

Proposition 3.2 (Density from summable increments)**.** *Under* (4) *and* (5)*,*

$$\lim_{n \to \infty} \frac{f(n)}{n} = L_{\text{jump}} := \frac{1}{3} \sum_{j \geq 0} \frac{\Delta_j}{j+1}. \tag{6}$$

Proof. For each fixed j, the asymptotic for C implies

$$\frac{C(\lfloor n/(j+1) \rfloor)}{n} \longrightarrow \frac{1}{3(j+1)}.$$

For all $n \geq 1$, the absolute value of the jth summand in the normalized expression of Lemma 3.1 is at most $|\Delta_j|/(j+1)$, because $0 \leq C(m) \leq m$. Extend the finite sum by zeros for $j \geq n$ and apply dominated convergence for series, using (5). ☐

This proves existence of the limit by the route taken in the submitted artifact. No numerical convergence experiment is needed.

4. The explicit coefficient expansion

The next formal identity reorganizes the increments at their arithmetic endpoints. It is useful because powers of two and three then appear explicitly in all denominators.

For $k \geq 0$, define

$$x_k = 4^k, \quad y_k = 3^{\lfloor \log_3(4^k) \rfloor}, \qquad u_k = 3^k, \quad v_k = 4^{\lfloor \log_4(3^k) \rfloor}.$$

The coefficient a_k is chosen by the first applicable row in the left table, and b_k by the first applicable row in the right table:

Condition	a_k	Condition	b_k
$15x_k < 16y_k$	78	$3u_k < 4v_k$	30
$9x_k < 10y_k$	30	$5u_k < 8v_k$	35
$3x_k < 4y_k$	-30	$3u_k < 5v_k$	29
$2x_k < 3y_k$	30	$u_k < 2v_k$	24
$x_k < 2y_k$	0	otherwise	-60
$3x_k < 8y_k$	-60		
otherwise	-12		

The inequalities are strict; a coefficient on a boundary is determined by continuing to the next row. These tables transcribe `coefficient4` and `coefficient3` in the accepted file.

Proposition 4.1 (Coefficient identity supplied by the source)**.**

$$L_{\text{jump}} = L_{\text{explicit}} := \frac{1}{180}\left(32 + \sum_{k\geq 0} \frac{a_k}{4^k} + \sum_{k\geq 0} \frac{b_k}{3^k}\right). \tag{7}$$

The source proves this in `jumpDensity_eq_explicitDensity`. The identity includes the normalization 180 and the constant 32; neither is an empirically fitted parameter.

Both series converge absolutely, since $|a_k| \leq 78$ and $|b_k| \leq 60$. If each is truncated after $k = N-1$, the total error in (7) is at most

$$\frac{1}{180}\left(104 \cdot 4^{-N} + 90 \cdot 3^{-N}\right). \tag{8}$$

Indeed, $\sum_{k\geq N} 78/4^k = 104 \cdot 4^{-N}$ and $\sum_{k\geq N} 60/3^k = 90 \cdot 3^{-N}$. This is a rigorous truncation estimate; it does not imply irrationality. An infinite series of rational numbers can have a rational sum.

Lemma 4.2 (Shift identity)**.** *If (c_k) is bounded, $b > 1$, $S = \sum_{k\geq 0} c_k/b^k$, and $q \geq 0$, then*

$$(b^q - 1)S = b^q \sum_{k=0}^{q-1} \frac{c_k}{b^k} + \sum_{k\geq 0} \frac{c_{k+q} - c_k}{b^k}.$$

Proof. Split S at q, multiply by b^q, and subtract S. Boundedness and $b > 1$ justify all operations by absolute convergence. □

The arithmetic part uses this identity along returns of the rotation associated with the relative powers of four and three. Establishing the resulting estimates, including nonvanishing, is a separate and substantial part of the submitted proof.

5. Rationality forces impossible approximations

Let
$$\mathcal{U}_{2,3} = \{2^a 3^b : a, b \in \mathbb{Z}\} \subseteq \mathbb{Q}_{>0}.$$
For a nonzero rational vector x, let $H(x)$ denote its multiplicative projective height: clear denominators, divide the resulting integer vector by the gcd of its coordinates, and take the maximum absolute coordinate. This agrees with the rational height used in the source.

Proposition 5.1 (Return-approximation interface)**.** *If L_{explicit} were rational, there would exist a fixed integer $4 \le m \le 404$, fixed coefficients $d_1, \dots, d_m \in \mathbb{Q}$, and sequences $P_r \in \mathbb{Z}$, $u_{r,i} \in \mathcal{U}_{2,3}$, $s_r, T_r \in \mathbb{R}$ with $s_r \to \infty$, such that eventually*
$$T_r \ge s_r, \qquad u_{r,i} \ge e^{-T_r}, \tag{9}$$
$$H(P_r, u_{r,1}, \dots, u_{r,m}) \le e^{27 s_r}, \tag{10}$$
$$0 < \left| \sum_{i=1}^{m} d_i u_{r,i} - P_r \right| \le e^{-T_r - s_r/200}. \tag{11}$$

This is the usable content of `rational_density_fixed_return_approximation`. The formal record contains additional quotient-height information, but (9)–(11) suffice for the final contradiction below. In particular, the dimension and coefficients are fixed before r varies.

The source first constructs return approximations whose coordinate labels may vary. It bounds their number and places the coefficients in a finite set, then passes to a subsequence and reindexes to fix the coefficient pattern. It also proves that the resulting errors are nonzero. This last step cannot be discarded: an exact zero would not contradict a lower bound for nonzero forms.

Proposition 5.2 (Arithmetic rigidity supplied by the source)**.** *For fixed $m \le 404$ and $d_1, \dots, d_m \in \mathbb{Q}$, there exists $c > 0$ such that, whenever $P \in \mathbb{Z}$, $u_i \in \mathcal{U}_{2,3}$, $0 < t \le 1$, $u_i \ge t$, and $E = \sum_i d_i u_i - P \ne 0$,*
$$|E| \ge c\, t\, H(P, u_1, \dots, u_m)^{-1/10800}. \tag{12}$$

This is `oneForm_integer_rigidity` with the source's `oneFormExponent`. Its proof uses a finite rational-hyperplane cover for sufficiently small forms at the ordinary, 2-adic, and 3-adic places. The covering theorem is established inside the submission in `rational_three_place_subspace_cover`. Reindexing specializes it to the one-form covering statement. Induction on dimension then eliminates coordinates on the exceptional hyperplanes and supplies a uniform positive constant for the fixed coefficient vector. The submitted source contains the full covering and elimination arguments; they are inputs to this exposition's short final deduction, not extra assumptions in its accepted target.

Final contradiction and proof of Theorem 1.1. Suppose L_{explicit} is rational, and take the data of Proposition 5.1. For large r, $s_r \ge 0$ and $T_r \ge 0$, so $t = e^{-T_r}$ lies in $(0, 1]$. Apply Proposition 5.2. Since $27/10800 = 1/400$, the height bound gives
$$H(P_r, u_{r,1}, \dots, u_{r,m})^{-1/10800} \ge e^{-s_r/400}.$$
Combining the lower and upper bounds on the same nonzero error yields
$$c\, e^{-T_r - s_r/400} \le e^{-T_r - s_r/200}.$$
Canceling the positive exponential factor gives
$$c \le e^{-s_r/400} \longrightarrow 0,$$
contradicting $c > 0$. Hence L_{explicit} is irrational. Propositions 3.2 and 4.1 identify it with the limit of $f(n)/n$, completing the deduction. ∎

6. Formal source guide and verification

The complete accepted Lean source and its public result and review are available through Conjectures. The following line numbers refer to the exact public submission body, not to a compiler-generated wrapper file.

Declaration or component	Line
`forkFree_iff`	24
`coefficient4`, `coefficient3`	215, 226
`explicitDensity`	260
`smoothRankBound`, `badWindow`	936, 1109
`globalUpper`	1643
`f_eq_globalUpper`	50426
`signedRank`, `signedRankJump`	65918, 65984
`signedRank_jumps_summable`	66257
`tendsto_core_weighted_series`	66357
`jumpDensity_eq_explicitDensity`	67601
`rational_three_place_subspace_cover`	72187
`oneForm_integer_rigidity`	72609
`rational_density_fixed_return_approximation`	73896
`tendsto_f_jumpDensity`	73975
`no_fixedReturnApproximation_of_oneForm_cover`	74036
`Bounty.target`	74198

On 22 September 2026 the public source was retrieved and matched byte for byte to the accepted artifact retained with the review. Its SHA-256 is

`2ebc87b06386daf9720ac037996ce87eb760`
`dc58097cd098ec6620b1c7df92b6.`

The public verification report records successful statement comparison, task commitment checks, permitted-axiom checks, and Lean kernel acceptance in the production sandbox. The corresponding manual review is approved. Its permitted axioms are `propext`, `Quot.sound`, and `Classical.choice`. No fresh Lean replay or independent-kernel run was performed when preparing this exposition.

Some intermediate source comments describe still-conditional stages of the development. The final declaration supplies the previously proved three-place covering theorem to those interfaces. This is why the accepted conclusion has no remaining covering hypothesis. The formal source is the verified object; this shorter prose account and the historical dating are separately checked editorial material.

References

[1] R. K. Guy, *Unsolved Problems in Number Theory*, second edition, Springer, 1994, B24, p. 80. Digitized second edition.

[2] K. Lebensold, *A divisibility problem*, Studies in Applied Mathematics **56**, 291–294. Publisher record: June 1977; also cited as volume 1976/77.

[3] D. Davis, *Forbidden subgraphs in divisor graphs and an Erdős divisibility problem*, arXiv:2604.17613v1, 19 April 2026. The abstract explicitly leaves irrationality open.

[4] T. F. Bloom, Erdős Problem 1062, problem statement and references.

[5] Conjectures, Erdős 1062(ii): accepted submission and review, submitted by JenW1N; public source and report retrieved 22 September 2026.