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ERDŐS PROBLEM 108: HIGH CHROMATIC NUMBER WITH SIX-COLOUR C_4-FREE SUBGRAPHS

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Abstract. The accepted submission constructs, for every integer M, a finite graph G with $\chi(G) > M$ such that every C_4-free subgraph of G is six-colourable. Consequently the Erdős–Hajnal assertion recorded as Erdős Problem 108 is false, already at girth five and chromatic threshold seven. The construction passes from an ordered multipartite base graph to its arc graph. A set-valued colouring transfers a lower bound on chromatic number, while a branching decomposition bounds the chromatic number of every four-cycle-free subgraph by $4 + 2$. We explain the deterministic argument, the finite probabilistic base construction, and their correspondence with the public Lean proof.

1. The question and its negative answer

All graphs in the construction are finite, undirected, simple graphs. Their chromatic number is the minimum number of colours in a proper vertex colouring. Girth is the length of a shortest cycle, with infinity for an acyclic graph. A subgraph may delete both vertices and edges; it need not be induced.

Erdős Problem 108 asks whether, for every $r \geq 4$ and $k \geq 2$, there is a finite $f(k,r)$ such that every graph with chromatic number at least $f(k,r)$ has a subgraph with girth at least r and chromatic number at least k [2]. The case $r = 4$ is a theorem of Rödl.

Theorem 1.1. *For every $M \in \mathbb{N}$ there exists a finite nonempty graph G satisfying*

$$\chi(G) > M, \qquad H \subseteq G, \quad C_4 \not\subseteq H \implies \chi(H) \leq 6.$$

In particular, choose $r = 5$ and $k = 7$. Any proposed value of $f(7,5)$ is exceeded by a graph from the theorem, but every subgraph of girth at least five is six-colourable. This contradicts the asserted property. The statement does not depend on an infinite-graph convention.

1.1. **Historical date.** The problem record cites Erdős's 1971 paper [1]. The announcement's "55-year-old question" counts $2026 - 1971$ calendar years from that documented publication. It is not a claim that the problem was first proposed in 1971; the cited proceedings concern a 1969 conference.

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2. Arc graphs transfer a chromatic lower bound

Let F be a finite graph with a linear order on its vertices. Orient each edge from its smaller endpoint to its larger endpoint. Define

$$\mathrm{Arc}(F) = \{(u, v) : u < v,\ uv \in E(F)\}.$$

The *arc graph* $A(F)$ has this set as its vertex set. Two arcs (u, v) and (x, y) are adjacent when $v = x$ or $y = u$.

Lemma 2.1. *If $A(F)$ is m-colourable, then F is 2^m-colourable.*

Proof. Given an arc colouring with colour set $[m]$, assign to each base vertex v the set $S(v) \subseteq [m]$ of colours appearing on arcs leaving v. If $u < v$ and uv is an edge, the colour of (u, v) belongs to $S(u)$. It cannot belong to $S(v)$: an arc leaving v with that colour would be adjacent to (u, v). Hence $S(u) \neq S(v)$, giving a proper colouring with at most 2^m subsets. $\square$

3. Why a four-cycle-free subgraph needs only six colours

Let H be any spanning subgraph of $A(F)$. An arc $a = (u, v)$ is *branching* in H if it has at least two distinct neighbours of the form (v, w). Write B for the set of branching arcs. Let J_B be the base subgraph of F whose edges are precisely the arcs in B, with their directions forgotten.

Suppose the maximum forward degree of F is at most d.

Lemma 3.1. *If H is C_4-free, then $\Delta(J_B) \leq d + \binom{d}{2}$.*

Proof. At a base vertex v, at most d selected edges leave v. Each selected edge entering v corresponds to a branching arc and therefore has two distinct outgoing arc-neighbours at v. Choose one such pair. Two different incoming arcs cannot choose the same pair: together with the two outgoing arcs they would form a four-cycle in H. There are at most $\binom{d}{2}$ possible pairs. This bounds the number of selected edges entering v and proves the claim. $\square$

Lemma 3.2. *If every subgraph $J \subseteq F$ of maximum degree at most $d + \binom{d}{2}$ is a-colourable, every C_4-free subgraph of $A(F)$ is $(a + 2)$-colourable.*

Proof. First consider a spanning subgraph H. By Lemma 3.1, J_B has a proper a-colouring. Colour a branching arc (u, v) by the colour of u in J_B. Adjacent branching arcs concatenate, so their tails are joined by a selected base edge. This colours $H[B]$.

The nonbranching arcs induce a forest in the degeneracy sense: every nonempty set of them has a vertex of degree at most one. Indeed, choose an arc with the least tail in the set. None of its neighbours in the set can precede it in the arc orientation, and it has at most one forward neighbour because it is nonbranching. Repeatedly remove such a vertex and colour in reverse order with two colours. Using disjoint palettes for B and its complement gives $a + 2$ colours.

For a subgraph that also deletes vertices, first add back the missing vertices as isolated vertices, apply the spanning result, then restrict the colouring. Four-cycle-freeness is preserved by this operation. $\square$

4. A deterministic criterion for the base graph

Write $e_F(X)$ for the number of edges inside X, and $e_F(X, Y)$ for the number between disjoint vertex sets. Fix an integer $K \geq 4$ and put $\eta = 1/(2(K + 1))$.

Proposition 4.1. *Suppose every nonempty vertex set S has a partition $S = X \,\dot\cup\, Y \,\dot\cup\, Z$ such that*

$$e_F(Y) = 0, \qquad e_F(X, Y) \leq |X|, \tag{1}$$

$$|Z| < \frac{2|S|}{16K(K + 1)}, \qquad e_F(X) \leq (1 + \eta)|X|. \tag{2}$$

Then every subgraph $J \subseteq F$ with $\Delta(J) \leq K$ is four-colourable.

Proof. It suffices to show that every nonempty induced subgraph of J has a vertex of degree at most three. Otherwise some S has minimum degree at least four, so $e_J(S) \geq 2|S|$. Put $s = |S|$, $x = |X|$, $y = |Y|$, $z = |Z|$, $e = e_J(X)$ and $f = e_J(X, Y)$. Then

$$2s \leq e + f + Kz, \quad e \leq (1+\eta)x, \quad f \leq \min(x, Ky), \quad y \leq s - x.$$

The piecewise linear function $(1+\eta)x + \min(x, K(s-x))$ is maximised on $[0, s]$ at $x = Ks/(K+1)$, since $K \geq 4$. Therefore

$$e + f \leq \frac{(2+\eta)K}{K+1} s < \left(2 - \frac{1}{K+1}\right) s.$$

But $Kz < s/(8(K+1))$. Combining the last two inequalities contradicts $2s \leq e + f + Kz$. Greedy colouring now gives four colours. □

5. The multipartite probability space

Fix $q \geq 1$ and set

$$C = 64q^2, \qquad A = 16CK(K+1), \qquad D = 2(K+1).$$

Take ordered independent parts $V_0, \ldots, V_{C-1}$, of positive sizes $n_0, \ldots, n_{C-1}$. For each vertex in V_i and each $j > i$, independently choose one uniformly random neighbour in V_j. The resulting graph F has forward degree at most $C - 1$. Its probability space is finite; the formal proof uses uniform counting.

5.1. Excluding low chromatic number.

For a vertex set I define its part densities and total weight by

$$p_i(I) = \frac{|I \cap V_i|}{n_i}, \qquad w(I) = \sum_i p_i(I).$$

The weights of all colour classes in any $2q$-colouring sum to C. Thus it is enough that every independent set have weight less than $C/(2q) = 32q$.

If $0 \leq p_i \leq 1$ and $\sum_i p_i \geq 32q$, discard entries less than $1/(4q)$. Their total is at most $C/(4q) = 16q$. The retained entries have total at least $16q$; the first retained entry therefore has a tail sum at least $16q - 1 \geq 15q$. Hence some index i satisfies

$$p_i \geq \frac{1}{4q}, \qquad \sum_{j>i} p_j \geq 15q.$$

For a fixed set with this property, independence requires each selected vertex in V_i to avoid the prescribed sets in later parts. Consequently

$$\mathbb{P}(I \text{ independent}) \leq \prod_{j>i} (1 - p_j)^{|I \cap V_i|} \leq \exp\left(-|I \cap V_i| \sum_{j>i} p_j\right) \leq e^{-3n_i}.$$

Choose sizes with $Cn_j \leq n_i$ whenever $i < j$, and $n_i \geq 8C$. The suffix from part i then contains at most $2n_i$ vertices. A union bound over its subsets and over i bounds the probability of a heavy independent set by

$$\sum_{i<C} 2^{2n_i} e^{-3n_i} \leq \frac{1}{8}.$$

Outside this event F is not $2q$-colourable.

5.2. **Excluding small dense sets in earlier parts.** The second bad event is the existence of an index i and a nonempty set $X \subseteq \bigcup_{j<i} V_j$ such that

$$|X| \leq An_i, \qquad e_F(X) \geq (1+1/D)|X|.$$

The source proves the following finite counting estimate. If the earlier parts contain at most N vertices, each has size at least L, and $3x \leq L$, then the probability that some x-set in those parts has at least $m = \lceil (1+1/D)x \rceil$ edges is at most

$$\binom{N}{x}\binom{x^2}{m}L^{-m}. \tag{3}$$

Indeed, choose the vertices and m prospective edges. A prescribed edge set has probability at most L^{-m}: compatible edges specify distinct random choices; incompatible choices give probability zero. Summing these probabilities gives (3).

Using $\binom{u}{v} \leq (3u/v)^v$, the expression is at most δ^x whenever

$$\frac{3^{2D+1}N^D x}{L^{D+1}} \leq \delta^D, \qquad \delta = \frac{1}{16C}. \tag{4}$$

The integer rounding is included in the source's `sparse_binomial_bound`. Summing δ^x over positive sizes and all C parts bounds this second bad event by $1/8$.

5.3. **Explicit choices of sizes.** Both estimates can be made simultaneous. The source chooses

$$R = (16C)^D 3^{2D+1}A + 3A + 8C + 2,$$
$$b = D+2, \qquad T = b^{C+1}, \qquad n_i = R^{T-b^{i+1}} \quad (0 \leq i < C).$$

These sizes decrease, satisfy $Cn_j \leq n_i$ for $i < j$, and exceed $8C$. For $i > 0$ put $L = n_{i-1}$ and $N = R^T$. The earlier parts have size at least L and total size at most N. If $x \leq An_i$, then $3x \leq L$. The exponent identity gives

$$RN^D n_i \leq L^{D+1}.$$

Since $R \geq (16C)^D 3^{2D+1}A$, this proves (4). The union of the two bad events has probability at most $1/4 < 1$, so there is a choice of F avoiding both.

5.4. **Obtaining the required cuts.** For any nonempty S, choose the last part i for which $|S| \leq An_i$. Partition S into its vertices before, in, and after that part, denoted X, Y, Z. The first part always qualifies, because $|S| \leq Cn_0 \leq An_0$. The set Y is independent and every vertex of X has at most one neighbour in Y. Avoiding the second bad event gives $e_F(X) \leq (1+1/D)|X|$. If there is a next part, its size satisfies $An_{i+1} < |S|$, hence

$$|Z| \leq Cn_{i+1} < \frac{|S|}{16K(K+1)}.$$

If there is no next part, Z is empty. Thus (1)–(2) hold. Proposition 4.1 shows that every base subgraph of maximum degree at most K is four-colourable.

6. Completing the counterexample

Given M, take

$$q = 2^M + 2, \quad C = 64q^2, \quad d = C - 1, \quad K = d + \binom{d}{2}.$$

The base construction gives a finite graph F with $\chi(F) > 2q$ and forward degree at most d. If $A(F)$ were M-colourable, Lemma 2.1 would give $\chi(F) \leq 2^M < 2q$, a contradiction. Thus $\chi(A(F)) > M$. Every C_4-free subgraph of $A(F)$ is six-colourable by Lemma 3.2 with $a = 4$. This proves Theorem 1.1.

7. Formal source and verification record

The public submission was verified on 15 September 2026 and approved in review on 16 September. The public result and its report record statement comparison, permitted-axiom checks and Lean kernel acceptance [3]. Preparing this exposition did not rerun Lean. The downloaded public proof is byte-identical to the retained accepted source. Its SHA-256 is

`a0b65a62937486813884b5bbc9fb36a2bb6`
`da42850a898126eddaecd5aaa0fa2`

Accepted source declaration	Line
`base_colorable_of_arc_colorable`	53
`selected_degree_le`	166
`nonbranching_colorable`	259
`arc_colorable_bound`	284
`colorable_four_of_sparse_cuts`	481
`model_badColor_probability_le`	1221
`model_badSparse_probability_le`	1412
`model_sizes_exist`	1621
`model_base_exists`	1730
`arc_counterexample_family`	1765
`counterexample_family`	1798
`Bounty.target`	1845

The final formal theorem negates the exact committed target. The construction is finite even though the formal result is transported to arbitrary universes. The accepted source refers to an attachment in two comments; the public account credit is retained without inferring an unconfirmed author list or publication-priority claim.

References

[1] P. Erdős, *Some unsolved problems in graph theory and combinatorial analysis*, Combinatorial Mathematics and its Applications (Proc. Conf., Oxford, 1969), 1971, pp. 97–109. Bibliographic record.
[2] T. F. Bloom, *Erdős Problem 108*, `https://www.erdosproblems.com/108`, accessed 17 September 2026.
[3] Conjectures, *Erdős 108: accepted submission by JenW1N*, 2026. Result and review; complete Lean proof; verification report.