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ERDŐS PROBLEM 14: A UNIFORM SQUARE-ROOT OBSTRUCTION TO UNIQUE SUMS

LIAM KRUER AND JENSEN KOHLMAYER

ABSTRACT. For a set A of nonnegative integers, let $E_A(N)$ count the integers from 1 to N that cannot be represented in exactly one unordered way as a sum of two elements of A , with equal summands allowed. We prove the uniform bound $E_A(N) > \sqrt{N}/12000$ for every $N \geq 2 \cdot 10^{24}$. This answers part (i) of Erdős Problem 14 affirmatively and part (ii) negatively. The proof combines a symmetric triple-moment identity, bounds for pairs in an arithmetic progression, and two incompatible estimates for a finite generating function. Both formal submissions share the same finite obstruction. We give that argument and the deductions of both parts, with links and a map to the accepted Lean declarations.

1. STATEMENT AND CONVENTIONS

Write $\mathbb{N} = \{0, 1, 2, \dots\}$. For $A \subseteq \mathbb{N}$, define

$$r_A^*(n) = |\{(a, b) \in A^2 : a \leq b, a + b = n\}|, \quad E_A(N) = |\{1 \leq n \leq N : r_A^*(n) \neq 1\}|.$$

Every pair in this definition is unordered: swapping two distinct summands does not produce a second representation. A diagonal pair (a, a) is allowed. Thus an exception is either a missing sum or a sum with at least two unordered representations. Zero is not counted by E_A , although zero may belong to A .

Theorem 1.1. *For every $A \subseteq \mathbb{N}$ and every integer $N \geq 2 \cdot 10^{24}$,*

$$E_A(N) > \frac{\sqrt{N}}{12000}. \tag{1}$$

The constants are explicit constants from the accepted formal proof; no optimality is asserted. In particular, for every $\varepsilon > 0$, $N^{1/2-\varepsilon} = O(E_A(N))$, and there is no set A for which $E_A(N) = o(\sqrt{N})$. These are respectively the positive answer to part (i) and the negative answer to part (ii).

1.1. Historical context and source of the proof. Erdős's 1992 paper [1], printed pages 43–44, defines the same count of missing or nonunique sums and explicitly conjectures that it can be $o(\sqrt{x})$. He describes earlier work with Nathanson and a subsequent rediscovery with Sós and Sárközy. Consequently, 1992 is a documented date for the question, not a claim about its first proposal. The modern two-part formulation is recorded as Erdős Problem 14 [2].

Date: Working draft — 16 September 2026.

Key words and phrases. Additive representation functions, unique sums, Erdős problems, generating functions, Lean.

Working prose exposition prepared by Conjectures from two accepted Lean submissions; this is not an author-approved journal publication.

The two Lean submissions were accepted on 14 September 2026 and approved in review on 15 September 2026 [3, 4]. This exposition follows their common finite argument and then explains each conclusion. Its prose has not itself been formally checked. No new Lean run was performed to prepare this document; verification refers to the recorded accepted submissions.

2. PAIR COUNTING AND THE TRIPLE-MOMENT IDENTITY

Let $S \subseteq \mathbb{N}$ be finite. We use the ordered counts

$$\begin{aligned} r_S(n) &= |\{(a, b) \in S^2 : a + b = n\}|, \\ d_S(n) &= |\{b \in S : 2b = n\}|, \\ w_S(n) &= |\{(a, b) \in S^2 : a + 2b = n\}|. \end{aligned}$$

Let $\mathcal{T}_S(n) = \{(a, b, c) \in S^3 : a + b + c = n\}$ and $\mathcal{W}_S(n) = \{(a, b) \in S^2 : a + 2b = n\}$. Say that n is *unique for S* when it has exactly one unordered representation by elements of S . In that case

$$r_S(n) + d_S(n) = 2. \quad (2)$$

Indeed, a nondiagonal representation contributes two ordered pairs and no diagonal, whereas a diagonal representation contributes one of each.

Lemma 2.1 (Pair bounds). *Let E be a finite exceptional set, $m = |E|$, and suppose $r_S(n) \leq 2$ for every $n \notin E$. If $S \subseteq [0, X]$ and $k = |S|$, then*

$$k^2 \leq 4X + 2 + mk. \quad (3)$$

If, additionally, all elements of S have the same residue modulo a positive integer q , then

$$k^2 \leq 4\lfloor X/q \rfloor + 2 + mk. \quad (4)$$

Proof. For every n , one has $r_S(n) \leq k$: its first coordinate determines its second. All ordered pairs have sums in $[0, 2X]$. Summing the bound 2 outside E and the bound k on E gives (3), with harmless overcounting on the exceptional sums. For the second assertion, write each element as $qj + r$ with $0 \leq j \leq \lfloor X/q \rfloor$. There are at most $2\lfloor X/q \rfloor + 1$ possible sums. The same counting argument gives (4). $\square$

Lemma 2.2 (Clean triple moment). *Suppose $S \subseteq [0, n]$ and $n - a$ is unique for S for every $a \in S$. Put $H = \sum_{a \in S} a$, $k = |S|$, and $B = \sum_{(a,b) \in \mathcal{W}_S(n)} b$. Then*

$$3H + 3B = n(k + w_S(n)). \quad (5)$$

Proof. For any weight $f : S \rightarrow \mathbb{R}$, summing (2) at $n - a$ gives

$$\sum_{(a,b,c) \in \mathcal{T}_S(n)} f(a) + \sum_{(a,b) \in \mathcal{W}_S(n)} f(a) = 2 \sum_{a \in S} f(a).$$

With $f = 1$, this gives $|\mathcal{T}_S(n)| + w_S(n) = 2k$. With $f(a) = a$, it gives $U + V = 2H$, where U and V are the first-coordinate sums over $\mathcal{T}_S(n)$ and $\mathcal{W}_S(n)$. Permutation symmetry of the three coordinates gives $3U = n|\mathcal{T}_S(n)|$. Also $V + 2B = nw_S(n)$. Eliminating U, V and $|\mathcal{T}_S(n)|$ yields $6H = 2n(k + w_S(n)) - 6B$, as required. $\square$

In particular, a clean target with $w_S(n) = 0$ satisfies $3H = nk$. When $w_S(n) = 1$, its unique weighted pair has second coordinate $b \in S$, and $3H + 3b = n(k + 1)$. These two identities control the targets where the weighted representation count is small.

3. A FINITE OBSTRUCTION

Theorem 3.1. *Let $T \geq 10^6$ be an integer and $X = T^4$. Suppose $S \subseteq [0, X]$ and $E \subseteq \mathbb{N}$ is finite, with*

- (1) $r_S(n) \leq 2$ for every $n \notin E$;
- (2) every $0 \leq n \leq X$ outside E is unique for S .

Then $T^2 < 1000|E|$.

Notice the difference between the hypotheses: uniqueness is required only up to X , but the upper bound on ordered representations is required at all nonexceptional sums. We will supply exceptions up to $2X$ when applying the theorem to a set A .

Assume for a contradiction that

$$m = |E| \leq T^2/1000. \quad (6)$$

Write $k = |S|$. By (3), $k^2 \leq 4T^4 + 2 + mk$, so

$$k \leq m + 2T^2 + 1 \leq 3T^2. \quad (7)$$

Here we used the elementary implication $u^2 \leq L^2 + mu \Rightarrow u \leq m + L$ for nonnegative u, m, L .

3.1. Few targets can have at most one weighted representation. Set

$$D = |\{0 \leq n \leq X : w_S(n) \leq 1\}|, \quad \mathcal{B} = S + E = \{a + e : a \in S, e \in E\}.$$

The dirty-target set $\mathcal{B}$ has size at most km . For $n \leq X$ let $S_n = S \cap [0, n]$ and $K_n = |S_n|$. If $n \notin \mathcal{B}$, then every $n - a$, $a \in S_n$, lies outside E and is unique for S_n . Nonnegativity of all summands ensures that restricting from S to S_n does not change any representation of a number at most n .

The coverage hypothesis also gives

$$n + 1 \leq K_n^2 + m \quad (0 \leq n \leq X), \quad (8)$$

because every integer in $[0, n] \setminus E$ is a sum of two elements of S_n . If $n > \lfloor X/1000 \rfloor$, then (6) and (8) imply $K_n > 0$ and

$$T^2 \leq 32(K_n + 1). \quad (9)$$

For example, $K_n^2 > (T^4 - T^2)/1000 \geq T^4/1024$ at our range of T , which is more than sufficient.

Group the clean targets in this range with $w_S(n) \leq 1$ by the value $K_n = K$. Since the prefixes are nested, equal cardinality means equal sets: within one group write $S_n = P$ and $H = \sum_{a \in P} a$. Also $w_P(n) = w_S(n)$. By Lemma 2.2, a target with $w_S(n) = 0$ satisfies $3H = nK$, so there is at most one such target in the group.

For the targets with $w_S(n) = 1$, choose the second coordinate b_n of the unique weighted pair. With $q = K + 1$,

$$3H + 3b_n = nq. \quad (10)$$

Thus $n \mapsto b_n$ is injective and the integers $3b_n$ all have the same residue modulo q . Let d be the number of these targets. The set $\{3b_n\}$ lies in $[0, 3X]$ and inherits the ordered-pair bound outside the dilated exceptional set $3E$. Applying (4) gives

$$d^2 \leq 4\lfloor 3X/q \rfloor + 2 + md.$$

By (9), $3X/q \leq 96T^2$. Hence $d^2 \leq (20T)^2 + md$, and consequently $d \leq m + 20T$. Each group therefore has at most $1 + m + 20T$ low targets.

There are at most $k + 1$ possible prefix cardinalities. Adding the small targets and the dirty targets gives

$$\begin{aligned} D &\leq \lfloor X/1000 \rfloor + 1 + km + (k + 1)(1 + m + 20T) \\ &\leq \frac{7}{1000}T^4 + 60T^3 + \left(3 + \frac{1}{1000}\right)T^2 + 20T + 2 \leq \frac{X}{100}. \end{aligned} \quad (11)$$

The final inequality follows upon division by T^4 and $T \geq 10^6$. Thus all but at most $X/100$ of the targets up to X have at least two weighted representations.

3.2. Incompatible generating-function estimates. Define the finite polynomial $F(z) = \sum_{a \in S} z^a$. For $0 \leq z < 1$, the ordered-pair bound gives

$$F(z)^2 = \sum_n r_S(n) z^n \leq \frac{2}{1-z} + mk. \quad (12)$$

Indeed, each nonexceptional coefficient is at most two, each exceptional coefficient at most k , and $z^n \leq 1$. On the other hand, $F(z)F(z^2) = \sum_n w_S(n) z^n$. Discarding terms with $n \geq X$ and allowing a loss of at most two for each low target yields

$$2 \sum_{n=0}^{X-1} z^n \leq F(z)F(z^2) + 2D. \quad (13)$$

Take $z = 1 - 4/X$. Since $X \geq 1000$, $0 < z < 1$ and

$$z^{-X} \geq (1 + 4/X)^X \geq 5,$$

using $(1 - u)^{-1} \geq 1 + u$ and Bernoulli's inequality. Thus $z^X \leq 1/5$, and

$$2 \sum_{n=0}^{X-1} z^n = \frac{X}{2} (1 - z^X) \geq \frac{2X}{5}. \quad (14)$$

Meanwhile (6) and (7) give $mk \leq 3X/1000$. Using (12) at z and z^2 , we obtain

$$\begin{aligned} F(z)^2 &\leq X/2 + 3X/1000 \leq 51X/100, \\ F(z^2)^2 &\leq X/4 + \frac{X}{2(X-2)} + 3X/1000 \leq 26X/100. \end{aligned}$$

The identity $2/(1 - z^2) = X/4 + X/(2(X-2))$ and the bound $X/(2(X-2)) \leq 1$ justify the second line. Therefore

$$F(z)F(z^2) \leq \frac{\sqrt{51 \cdot 26}}{100} X < \frac{37X}{100}. \quad (15)$$

Equations (11), (13), (14), and (15) now imply

$$\frac{40X}{100} \leq F(z)F(z^2) + 2D < \frac{37X}{100} + \frac{2X}{100} = \frac{39X}{100},$$

a contradiction. This proves Theorem 3.1.

4. FROM THE FINITE OBSTRUCTION TO BOTH PARTS

Fix any $A \subseteq \mathbb{N}$ and $T \geq 10^6$. Take

$$X = T^4, \quad S = A \cap [0, X], \quad E = \{0 \leq n \leq 2X : r_A^*(n) \neq 1\}.$$

For $n \leq X$ outside E , the unique representation lies entirely in S . For $n \leq 2X$ outside E , restricting to S leaves at most two ordered representations; for $n > 2X$, there are none. Thus the two hypotheses of Theorem 3.1 hold. As zero contributes at most one additional exception,

$$T^2 < 1000|E| \leq 1000(E_A(2T^4) + 1). \quad (16)$$

This applies equally to finite and infinite sets A .

4.1. Part (i): an affirmative answer. Let $N \geq 2 \cdot 10^{24}$ and choose the integer $T = \lfloor (N/2)^{1/4} \rfloor$. Then $T \geq 10^6$ and

$$2T^4 \leq N < 2(T+1)^4 \leq 32T^4.$$

The formal proof makes this choice using two integer square roots. By monotonicity of E_A and (16), $T^2 < 1000(E_A(N) + 1)$. Since $T^2 \geq 2000$, this first forces $E_A(N) > 1$, and hence

$$T^2 < 2000E_A(N).$$

Also $\sqrt{N} < \sqrt{32}T^2 \leq 6T^2$. Consequently $\sqrt{N} < 12000E_A(N)$, proving Theorem 1.1. For any $\varepsilon > 0$ and $N \geq 1$, $N^{1/2-\varepsilon} \leq \sqrt{N}$; thus

$$N^{1/2-\varepsilon} = O(E_A(N)).$$

In fact the constant 12000 and the threshold above work uniformly in A and ε .

4.2. Part (ii): a negative answer. Theorem 1.1 immediately rules out $E_A(N) = o(\sqrt{N})$. The separate accepted part-(ii) submission obtains the contradiction directly from (16), without depending on the part-(i) submission. If $E_A(N) = o(\sqrt{N})$, then for all sufficiently large N ,

$$E_A(N) \leq \frac{\sqrt{N}}{4000}.$$

Choose a sufficiently large integer $T \geq 10^6$ and put $N = 2T^4$. Since $\sqrt{2T^4} \leq 2T^2$, this gives $1000E_A(2T^4) \leq T^2/2$. Equation (16) then yields

$$T^2 < T^2/2 + 1000 \leq T^2,$$

again impossible. Thus no set $A \subseteq \mathbb{N}$ has the proposed little- o exception count.

5. FORMAL PROOF RECORDS AND CORRESPONDENCE

The public record for each part links the complete accepted Lean source and the recorded verification checks:

- Part (i), proof: [result and review](#); [Lean source](#).
- Part (ii), refutation: [result and review](#); [Lean source](#).

Both production reports record Lean kernel acceptance and agreement with the committed statement. The optional independent kernel was not run. The permitted axioms were `propext`, `Quot.sound`, and `Classical.choice`. Both sources contain the common finite obstruction and its transfer to arbitrary sets. Their final deductions are separate.

The table uses declaration names in namespace `Erdos14Verification`; line numbers refer to the accepted part-(i) file, except for the final row.

Mathematical step	Lean declaration	Line
Diagonal convention	<code>unique_pairs_card_add_diagonal</code>	70
Triple-moment identity	<code>clean_triple_moment</code>	200
Interval pair bound	<code>interval_pair_count</code>	260
Progression pair bound	<code>progression_pair_count</code>	272
One-target class bound	<code>linear_targets_pair_bound</code>	409
Prefix coverage	<code>prefix_card_lower</code>	519
Fixed-prefix low targets	<code>fixed_prefix_low_bound</code>	612
Low-target density	<code>low_targets_density</code>	683
Generating upper bound	<code>generating_upper</code>	804
Generating lower bound	<code>generating_lower</code>	844
Finite obstruction	<code>finite_obstruction</code>	971
Transfer to an arbitrary set	<code>set_scale_obstruction</code>	1115

Mathematical step	Lean declaration	Line
Uniform bound	<code>uniform_sqrt_bound</code>	1183
Part (i)	<code>part_i_positive</code>	1206
Part (ii), in its own file	<code>part_ii_negative</code>	1132

REFERENCES

- [1] P. Erdős, *Some of my forgotten problems in number theory*, Hardy–Ramanujan Journal **15** (1992), 34–50, especially 43–44. [Journal record](#) and [original paper](#).
- [2] T. F. Bloom, *Erdős Problem #14*, <https://www.erdosproblems.com/14>.
- [3] Liam Kruer and Jensen Kohlmeier, *Erdős Problem 14, part (i)*, accepted Lean submission, Conjectures, 14 September 2026; [public formal proof](#).
- [4] Liam Kruer and Jensen Kohlmeier, *Erdős Problem 14, part (ii)*, accepted Lean refutation, Conjectures, 14 September 2026; [public formal proof](#).