Conjectures.io

Formal proof verified through Conjectures

ADDITIVE COMPLETENESS OF TWO BINARY FLOOR SEQUENCES ERDŐS PROBLEM 354(I): A MATHEMATICAL RECONSTRUCTION OF A LEAN PROOF

Formal proof credited to JenW1N

Abstract. For $\alpha, \beta > 0$ with irrational ratio, the cited formal submission proves that every sufficiently large integer is a sum of finitely many indexed terms from the two sequences $\lfloor 2^n\alpha \rfloor$ and $\lfloor 2^n\beta \rfloor$, with each index in each sequence used at most once. We develop the elementary arithmetic, the missing-sum joining obstruction, the exact binary-carry identities, the marked carry contraction and its quantitative amplification, elementary smoothing, tower generation and ergodicity, the zero-block displacement estimate, the Hilbert-space disjointness argument, and the final transport contradiction. The adaptive correlation and marked displacement estimates are derived with their finite-window boundary terms and exact constants. A declaration map connects the mathematical arguments to the supplied formal source.

1. The theorem and the meaning of a subset sum

Theorem 1 (The result of the cited formal submission)**.** *Let $\alpha, \beta > 0$ and $\alpha/\beta \notin \mathbb{Q}$. There exists an integer N such that every integer $z \geq N$ admits finite subsets $S, T \subset \mathbb{N}$ with*

$$z = \sum_{n \in S} \lfloor 2^n\alpha \rfloor + \sum_{n \in T} \lfloor 2^n\beta \rfloor. \tag{1}$$

The index set begins at zero. The two families of indexed terms are separate. Thus equal numerical values arising at different indices or in different sequences remain separate available summands. An index can occur only once within its own finite set. The theorem does not supply an explicit threshold N, and it does not assert uniqueness of representations.

Write

$$h_\alpha(n) = \lfloor 2^n\alpha \rfloor, \qquad d_\alpha(n) = h_\alpha(n+1) - 2h_\alpha(n).$$

For a sequence $a = (a_n)$ let

$$\Sigma(a) = \left\{ \sum_{n \in S} a_n : S \subset \mathbb{N} \text{ finite} \right\}.$$

The desired conclusion is that $\Sigma(h_\alpha) + \Sigma(h_\beta)$ contains every sufficiently large integer. In the source this is `CompletePair`. The final statement is expressed using the interleaved sequence with even positions $h_\alpha(n)$ and odd positions $h_\beta(n)$; the two formulations agree because the even and odd index sets are disjoint.

Date: 18 October 2026.

Expository manuscript prepared from the submitted solution. Mathematical exposition reconstructed from the formal submission. A separate agent reviewed the mathematical reconstruction, including the analytical estimates and final deductions, and found no substantive gap. This is not external peer review or a kernel check of the prose. The original Lean theorem environment has not been freshly built for this manuscript.

It suffices to prove the result for $\alpha, \beta \geq 1$. Choose M with $2^M\alpha, 2^M\beta \geq 1$. Then

$$h_{2^M\alpha}(n) = h_\alpha(M+n), \qquad h_{2^M\beta}(n) = h_\beta(M+n),$$

and the ratio is unchanged. A representation using these tail sequences is a representation using the original sequences. No summand is reused by the shift of indices.

2. Binary digits and the separation of finite sums

Lemma 2. *For every real α and $n \geq 0$, $d_\alpha(n) \in \{0,1\}$. Moreover*

$$h_\alpha(n) - \sum_{j<n} h_\alpha(j) = h_\alpha(0) + \sum_{j<n} d_\alpha(j). \tag{2}$$

If $\alpha \geq 1$, then $h_\alpha(n) \geq 2^n$ and $\sum_{j<n} h_\alpha(j) < h_\alpha(n)$.

Proof. Writing $2^n\alpha = k + x$ with $k \in \mathbb{Z}$ and $0 \leq x < 1$, the digit is $\lfloor 2x \rfloor$, hence zero or one. The height recurrence $h(n+1) = 2h(n) + d(n)$ gives (2) by induction. When $\alpha \geq 1$, $h(0) \geq 1$, so the right-hand side is positive, and the same recurrence proves $h(n) \geq 2^n$. □

Let $F_\alpha(n)$ be the finite set of subset sums using indices below n. Its recursion is

$$F_\alpha(0) = \{0\}, \qquad F_\alpha(n+1) = F_\alpha(n) \cup (h_\alpha(n) + F_\alpha(n)).$$

The two parts are disjoint by Lemma 2, and every element of $F_\alpha(n)$ lies in $[0, h_\alpha(n))$. Thus

$$|F_\alpha(n)| = 2^n. \tag{3}$$

In particular a single normalized sequence has unique finite subset-sum representations, although the theorem about the pair does not assert uniqueness.

Lemma 3. *The floor digit sequence has infinitely many zeros for every real parameter. If α is not dyadic rational, it also has infinitely many ones and infinitely many adjacent changes.*

Proof. If all digits from index N onward are zero, the recurrence gives $h_\alpha(N+k) = 2^k h_\alpha(N)$. Dividing the floor inequalities by 2^k and letting $k \to \infty$ yields $2^N\alpha = h_\alpha(N)$, so α is dyadic. If all those digits are one, the recurrence instead gives

$$h_\alpha(N+k) = 2^k(h_\alpha(N) + 1) - 1.$$

The lower floor inequality implies $2^N\alpha \geq h_\alpha(N) + 1$ in the limit, contradicting the strict upper floor inequality. Thus zeros occur arbitrarily far out. A sequence with only finitely many adjacent changes is eventually constant; either constant tail contradicts the conclusions just established when α is nondyadic. □

This argument fixes the terminating-expansion convention at dyadic parameters. It never replaces a terminating binary expansion by an infinite tail of ones.

3. A positive-density symbolic model and missing sums

Let $X = \{0,1\}^{\mathbb{Z}}$ with its compact product topology and shift

$$(\sigma^k x)(j) = x(j+k).$$

Define the subset-sum name

$$x_\alpha(j) = \mathbf{1}_{\Sigma(h_\alpha)}(j).$$

For normalized parameters, its negative coordinates vanish. Let $C = \{x \in X : x(0) = 1\}$, a clopen cylinder. A *name limit* of x_α is any weak limit along $N \to \infty$ of

$$\mu_{\alpha,N} = \frac{1}{N+1} \sum_{j=0}^{N} \delta_{\sigma^j x_\alpha}.$$

Such subsequential limits exist because the space of probability measures on the compact metrizable space X is compact. Every name limit is shift invariant: for continuous f the difference between its integrals against $\sigma_*\mu_{\alpha,N}$ and $\mu_{\alpha,N}$ is

$$\frac{f(\sigma^{N+1}x_\alpha) - f(x_\alpha)}{N+1},$$

which tends to zero.

Lemma 4. *Every name limit μ of x_α, with $\alpha \geq 1$, satisfies*

$$\mu(C) \geq \frac{1}{2\alpha} > 0.$$

Proof. For sufficiently large N, choose n with $h_\alpha(n) \leq N < h_\alpha(n+1)$. The 2^n distinct sums in $F_\alpha(n)$ lie in $[0, N]$, so

$$\mu_{\alpha,N}(C) \geq \frac{2^n}{N+1}.$$

Since $N+1 \leq h_\alpha(n+1) \leq 2^{n+1}\alpha$, the last quantity is at least $1/(2\alpha)$. The indicator of the clopen set C is continuous, so the bound passes to every weak limit. □

An *anti-joining* of shift-invariant measures μ, ν is a probability measure η on $X \times X$ with these marginals which is invariant under $(x, y) \mapsto (\sigma x, \sigma^{-1}y)$. Define $D(\alpha, \beta)$ to mean that every anti-joining between every pair of their name limits equals the product measure. This relation is symmetric: swap the coordinates, then use invariance under the inverse transformation.

Proposition 5. *If $\alpha, \beta \geq 1$ and $D(\alpha, \beta)$, then the pair is additively complete.*

Proof. If not, choose unrepresented integers $N_r \to \infty$. Define

$$\eta_r = \frac{1}{N_r + 1} \sum_{j=0}^{N_r} \delta_{(\sigma^j x_\alpha, \sigma^{N_r - j} x_\beta)}.$$

Its marginals are the two prefix empirical measures. It gives zero mass to $C \times C$, since a point of that cylinder would supply $j \in \Sigma(h_\alpha)$ and $N_r - j \in \Sigma(h_\beta)$. Extract a convergent subsequence. The telescoping endpoint argument shows its limit η is an anti-joining of name limits μ, ν. Clopenness gives $\eta(C \times C) = 0$, but Lemma 4 gives

$$(\mu \times \nu)(C \times C) = \mu(C)\nu(C) > 0.$$

Thus η is a nonproduct anti-joining, contradicting $D(\alpha, \beta)$. □

The obstruction is therefore concrete: infinitely many missing integers produce an invariant coupling that avoids simultaneous subset sums at the origin. Positive marginal densities alone do not prove completeness; the remaining argument excludes that coupling.

4. Exact return positions and binary carries

Fix $\alpha \geq 1$ and a stage m. Write $h_m = h_\alpha(m)$ and $d_j = d_\alpha(m + j)$. Define

$$R_m(q) = q h_m + \sum_{j \geq 0} d_j \left\lfloor \frac{q}{2^{j+1}} \right\rfloor. \tag{4}$$

The sum is finite. If $q = \sum_i \varepsilon_i 2^i$ is its binary expansion, repeated use of the height recurrence gives

$$R_m(q) = \sum_i \varepsilon_i h_\alpha(m + i). \tag{5}$$

For clarity, substituting

$$h_\alpha(m + i) = 2^i h_m + \sum_{j<i} 2^{i-1-j} d_j$$

into the right side makes the coefficient of d_j exactly $\lfloor q/2^{j+1} \rfloor$. Thus R_0 enumerates the subset-sum set in increasing order, because changing the highest different bit increases the sum by more than all possible changes below it. More quantitatively,

$$qh_m \le R_m(q) \le q(h_m+1), \qquad R_m(q+1) \ge R_m(q) + h_m. \tag{6}$$

The first upper bound uses $\sum_{j\ge 0} \lfloor q/2^{j+1} \rfloor \le q$, and the step bound follows because every floor in (4) is nondecreasing.

Let $c_j(x,q) \in \{0,1\}$ be the carry into binary position j when adding x and q; $c_0 = 0$. Equivalently

$$c_j(x,q) = \left\lfloor \frac{x+q}{2^j} \right\rfloor - \left\lfloor \frac{x}{2^j} \right\rfloor - \left\lfloor \frac{q}{2^j} \right\rfloor.$$

Subtracting (4) at $x+q, x, q$ proves the exact identity

$$R_m(x+q) - R_m(x) - R_m(q) = \sum_{j\ge 0} d_j c_{j+1}(x,q). \tag{7}$$

Its truncated cost is

$$C_{m,q,L}(x) = \sum_{j<L} d_j c_{j+1}(x,q).$$

For x uniform on $\{0,\dots,2^L-1\}$ and $j < L$, the probability of a carry into $j+1$ is

$$\frac{q \bmod 2^{j+1}}{2^{j+1}}.$$

Indeed the residue of x modulo 2^{j+1} is uniform, and precisely the last $q \bmod 2^{j+1}$ residues cause a carry. If $q \le 2^\ell$, then

$$\mathbb{E} C_{m,q,L} \le \ell + 1. \tag{8}$$

For $j < \ell$ bound the probability by one; for $j \ge \ell$ bound it by $2^{\ell-j-1}$ and sum the geometric tail. This logarithmic mean-cost bound, rather than a worst-case bound linear in L, controls the error terms later.

5. A marked carry forces Hilbert-space contraction

Let U be a surjective linear isometry of a real Hilbert space. A binary carry evolves by

$$c' = \mathbf{1}_{q_j + x_j + c \ge 2}.$$

It resets to q_j if $x_j = q_j$, preserves the incoming carry if $x_j = 1 - q_j$, and equals x_j if the incoming carry is $1 - q_j$. Suppose the three query bits are $(a, 1-a, e)$ and the middle spacer digit is one. Consider input bits $(a,0,e)$ and $(a,1,e)$. The first bit resets the carry to a. At the second bit, the output is respectively zero or one. The third bit resets both carries to e. Thus the outgoing states coincide, and their weighted carry costs differ by exactly one. Arbitrary fixed prefixes and tails preserve this cost difference.

Lemma 6. *If a uniform average over three input bits contains this marked pair, then for every vector f its norm is at most*

$$\frac{3}{4}\|f\| + \frac{1}{8}\|f + Uf\|. \tag{9}$$

This is strictly smaller than $\|f\|$ if $Uf \ne f$.

Proof. Six of the eight summands are isometric images of f and contribute at most $6\|f\|$ to the triangle inequality. The selected pair is $U^a f, U^{a+1} f$ for some common exponent, and its sum has norm $\|f + Uf\|$. Divide by eight. Since U is an isometry,

$$\|f + Uf\|^2 = 4\|f\|^2 - \|Uf - f\|^2.$$

If $Uf \ne f$, this is strictly below $4\|f\|^2$, proving strictness. □

Define the finite carry average

$$P_{m,q,L}f = 2^{-L}\sum_{x<2^L} U^{C_{m,q,L}(x)}f.$$

For fixed m, q, these averages converge as $L \to \infty$. To see this, extend a uniform input word by more bits. Its earlier cost is unchanged. A difference in the total cost requires a carry beyond the first L bits, whose probability is at most $q/2^L$ once $2^L > q$. Each resulting vector differs by at most $2\|f\|$, so the averages are Cauchy, with error bounded by $2q2^{-L}\|f\|$. Denote the limit by $P_{m,q}f$. It is a contraction, and the marked inequality passes to the limit whenever

$$q_k \neq q_{k+1}, \qquad d_\alpha(m+k+1) = 1.$$

A longer marked block can begin with a query-bit transition and delay its weighted cost until the next spacer digit one. Use equal reset bits at the ends, complementary input/query bits in between to preserve the two possible carry states, and choose the first subsequent spacer digit one so all earlier weights vanish. The two selected input words again have a common final carry and costs differing by one. If every block of H spacer digits contains a one, such a block has length at most $H+2$. Its selected pair has probability at least

$$\delta = 2 \cdot 2^{-(H+2)} > 0.$$

This is the combinatorial source of a uniform contraction weight.

6. The tower realization and its quantitative inputs

A rigorous passage from carry averages to symbolic correlations needs more information than the binary subset-sum name retains. The source therefore labels every integer simultaneously by its position in every return block. For $t \geq 0$, let $q_m(t)$ be the unique integer with

$$R_m(q_m(t)) \leq t < R_m(q_m(t)+1).$$

Its stage-m label is

$$\lambda_m(t) = \min\{t - R_m(q_m(t)), h_m\}.$$

The labels $0, \ldots, h_m - 1$ are ordinary tower levels; h_m groups the spacers outside the ordinary part. For $t < 0$ use the spacer label. The full labeled name lies in the compact space

$$Y_\alpha = \prod_{t\in\mathbb{Z}}\prod_{m\geq 0}\{0,\ldots,h_m\}.$$

Let ρ be any limit of the prefix empirical measures of this name. It is shift invariant. The projection $\lambda_0(t) = 0$ recovers the subset-sum name, by (5). Let B_m be the stage-m ordinary body and $B_{m,0}$ its base. Put $w_m = \rho(B_{m,0})$.

Ordinary levels have equal measure: the shift moves each ordinary level to the next, apart from an initial finite defect disappearing in the empirical limit. The disjoint levels therefore give $\rho(B_m) = h_m w_m$ and $w_m \leq 1/h_m$. For a prefix ending at N, there are $q_m(N)+1$ base positions. By (6),

$$N + 1 \leq (q_m(N)+1)(h_m+1),$$

so $w_m \geq 1/(h_m+1)$. Consequently

$$\rho(Y_\alpha \setminus B_m) \leq \frac{1}{h_m+1} \leq 2^{-m}, \qquad w_m \leq 2^{-m}. \tag{10}$$

These quantitative bounds are enough to make the later linear-width errors vanish.

Write $U_\alpha(t)f = f \circ \sigma^t$ on the real space $L^2(\rho)$. A tower observable is a function of one label $\lambda_m(0)$. We first prove the structural properties of these observables.

6.1. **Exact refinement and the passage to name limits.** The finite binary expansion in (5) gives the block identity

$$R_m(2^L q + r) = R_{m+L}(q) + R_m(r) \qquad (0 \le r < 2^L). \tag{11}$$

Indeed the low L bits contribute $R_m(r)$ and the remaining bits contribute $R_{m+L}(q)$. In particular $R_m(2^L) = h_{m+L}$. Together with the step bound in (6), this implies

$$R_m(r) + h_m \le h_{m+L} \qquad (0 \le r < 2^L).$$

Thus each ordinary stage-$(m+L)$ block contains exactly 2^L disjoint ordinary stage-m copies, starting at the offsets $R_m(r)$. There are no further stage-m bases in the spacers after the ordinary stage-$(m+L)$ block: its next stage-m base, by (11), is exactly the next stage-$(m+L)$ base.

For an ordinary high-stage label $0 \le i < h_{m+L}$ define $\phi_{m,L}(i) = \lambda_m(i)$, using the initial block, and set $\phi_{m,L}(h_{m+L}) = h_m$. The previous decomposition proves, for every integer t,

$$\lambda_m(t) = \phi_{m,L}(\lambda_{m+L}(t)). \tag{12}$$

For $t \ge 0$ write its high-stage block index as q. If its offset is ordinary, (11) identifies all lower bases with their offsets in the initial block; if its offset is a spacer, it is also a lower-stage spacer by the preceding paragraph. For $t < 0$ both sides are the prescribed spacer labels. More explicitly, for every ordinary lower level $0 \le i < h_m$ and every high-stage label $0 \le j \le h_{m+L}$,

$$\phi_{m,L}(j) = i \quad \Longleftrightarrow \quad j = R_m(r) + i \text{ for some } 0 \le r < 2^L. \tag{13}$$

The offsets on the right are distinct and below h_{m+L}.

We shall also use the following local translation rule:

$$\lambda_m(t) = i < h_m, \quad 0 \le i + s < h_m \quad \Longrightarrow \quad \lambda_m(t+s) = i + s. \tag{14}$$

This follows by writing $t = R_m(q) + i$; the whole interval from $R_m(q)$ through $R_m(q) + h_m - 1$ is inside that return block.

All the displayed relations concern only finitely many coordinates at a time. Their validity sets are therefore clopen in Y_α. Since they hold on every shift of the original labeled name, each such set has empirical measure one, and then measure one in every weak name limit. We may consequently use (12)–(14) almost everywhere in the invariant system. Countably many indices and shifts can be combined after discarding one null set. This argument establishes the transfer to the limit, rather than treating the limit configurations as literal translates of the original name.

Equal ordinary-level measures and (13) now give the exact base refinement

$$w_m = 2^L w_{m+L}. \tag{15}$$

Indeed an ordinary lower level is the disjoint union, up to a null set, of its 2^L specified higher levels, each of measure w_{m+L}. The same identity with $i = 0$ gives the base formula.

Proposition 7 (Tower generation and ergodicity)**.** *For every labeled-name limit ρ, tower observables over all stages are dense in $L^2(\rho)$ and refine compatibly from one stage to the next. Every vector fixed by $U_\alpha(1)$ is constant.*

Proof. Compatibility is exactly (12): the stage-k observable $a(\lambda_k(0))$ becomes $a(\phi_{k,n-k}(\lambda_n(0)))$ at any stage $n \ge k$, with the same uniform bound.

For density, take a cylinder function F depending on finitely many coordinates (t, k), say $|t| \le M$ and $k \le K$. For $n \ge K$ sufficiently large that $h_n > 2M$, suppose its stage-n label i satisfies

$$M \le i < h_n - M.$$

By (14) the stage-n label at time t is $i + t$, and by (12) every required lower label is $\phi_{k,n-k}(i + t)$. Thus the entire window on which F depends is determined by i. Define a stage-n observable F_n by

this value on the indicated interior levels, and put it equal to zero on the remaining labels. Almost everywhere, $F_n = F$ on the interior. The exceptional set has measure at most

$$2Mw_n + \rho(B_n^c) \le (2M+1)2^{-n}.$$

Hence $\|F - F_n\|_2 \le \|F\|_\infty \sqrt{(2M+1)2^{-n}} \to 0$. Cylinder functions are uniformly dense in the continuous functions on this countable product of finite alphabets: uniform continuity reduces any continuous function to a finite coordinate window. Continuous functions are dense in L^2 for a Borel probability measure on a compact metrizable space. This proves density of tower observables.

To prove the fixed-vector assertion, let $f \in L^2(\rho)$ satisfy $f \circ \sigma = f$. Let $\mathcal{P}_n$ be the finite partition into the stage-n labels and let $H_n = \mathbb{E}(f \mid \mathcal{P}_n)$, the orthogonal projection onto stage-n observables. The partitions refine by (12), and the established density implies $H_n \to f$ in L^2: every fixed earlier-stage approximation belongs to all sufficiently late projection spaces.

For adjacent ordinary levels, (14) in both directions identifies one level with the inverse shift of the next. Shift invariance of f and ρ therefore makes $\int_{\{\lambda_n = i\}} f \, d\rho$ the same for all $i < h_n$. Since those levels have equal positive measure, H_n has one constant value c_n throughout B_n. Moreover

$$c_n = \frac{\int_{B_n} f \, d\rho}{\rho(B_n)} \longrightarrow \int f \, d\rho,$$

using $\rho(B_n) \to 1$ and absolute continuity of the integral. On the single spacer atom B_n^c, conditional Cauchy–Schwarz gives

$$\int_{B_n^c} |H_n|^2 \, d\rho \le \int_{B_n^c} |f|^2 \, d\rho \longrightarrow 0.$$

(The inequality is interpreted as zero on both sides if this atom has zero measure.) Therefore H_n also converges in L^2 to the constant $\int f \, d\rho$. Uniqueness of the limit proves that f is constant. □

6.2. **The two-piece carry comparison.** Fix a stage m and write

$$h = h_m, \quad w = w_m, \quad e = \rho(B_m^c), \quad R = R_m, \quad \Delta = R(q+1) - R(q) - h.$$

Suppose $R(q) \le t < R(q+1)$ and $q + 1 \le 2^\ell$. Put $v = t - R(q)$, so $0 \le v < h + \Delta$. The spacer gap satisfies

$$0 \le \Delta \le \ell + 1. \tag{16}$$

For the upper bound, apply (7) to adding 1 to q. Since $q + 1 < 2^{\ell+1}$, carries above those $\ell + 1$ positions vanish; each remaining weighted carry is at most one. This identity gives exactly Δ.

Choose a higher-stage depth K, and put $N = 2^K$, $W = w_{m+K} = w/N$. For a uniform residue $0 \le r < N$, abbreviate

$$c_0(r) = C_{m,q,K}(r), \qquad c_1(r) = C_{m,q+1,K}(r), \qquad \gamma(r) = c_0(r) + \Delta + c_1(r).$$

Equations (8) and (16) give

$$\mathbb{E}_r \gamma(r) \le 3(\ell + 1). \tag{17}$$

For a source level $0 \le i < h$, choose the query $j(i) = q$ if $i + v < h$, and $j(i) = q + 1$ otherwise. Its predicted destination coordinate in an ordinary stage-m copy is

$$z_r(i) = \begin{cases} i + v - c_0(r), & i + v < h, \\ i + v - h - \Delta - c_1(r), & i + v \ge h. \end{cases} \tag{18}$$

At most $\gamma(r)$ source levels have $z_r(i) \notin [0, h)$. To verify this count, the first branch has $z_r(i) < h$ automatically, and a negative value requires $i + v < c_0(r)$, which occurs for at most $c_0(r)$ nonnegative integers i. In the second branch the upper bound follows from $i < h$ and $v < h + \Delta$; failure of the lower bound requires

$$0 \le i + v - h < \Delta + c_1(r),$$

an interval containing at most $\Delta + c_1(r)$ integers. We call these levels bad.

If $r+q+1<N$ and i is good, (7) has no contribution above position K and gives the exact coordinate identity

$$R(r)+i+t=R(r+j(i))+z_r(i). \tag{19}$$

Both the source and destination are ordinary levels inside the stage-$(m+K)$ tower. The local shift rule and exact refinement therefore identify the translated lower-stage label with $z_r(i)$. There is no probabilistic independence assumption here: this is an identity on each specified higher-level atom, transferred to the name limit as in Section 6.1.

Let f,g be stage-m observables bounded by $F,G\ge 0$. Write $L_i=\{\lambda_m(0)=i\}$ and $L_{r,i}^K=\{\lambda_{m+K}(0)=R(r)+i\}$. Their measures are w and W. Put

$$s_r(i)=z_r(i)-i=t-R(j(i))-C_{m,j(i),K}(r).$$

On each good cell with $r+q+1<N$, both integrands below are constant and (19) gives

$$\int_{L_{r,i}^K} g(f\circ\sigma^t)\,d\rho=N^{-1}\int_{L_i} g(f\circ\sigma^{s_r(i)})\,d\rho. \tag{20}$$

Indeed both sides equal $Wg(i)f(z_r(i))$, where $f(i),g(i)$ denote the values on ordinary labels.

On a bad cell each side has absolute value at most FGW, so its difference is at most $2FGW$. There are at most $q+1$ residues with $r+q+1\ge N$; for these top-crossing residues simply bound all h cells this way. The union of all source-copy levels is B_m up to a null set by (13). The remaining integral over B_m^c is at most FGe. Define the predicted correlation

$$J_K=\mathbb{E}_r\sum_{i=0}^{h-1}\int_{L_i} g(f\circ\sigma^{s_r(i)})\,d\rho.$$

Summing the preceding bounds, and allowing the bad-level contribution for all residues even where a top bound was used, proves

$$|\langle g,U_\alpha(t)f\rangle-J_K|\le FGe+2FG\big(3(\ell+1)w+(q+1)hW\big). \tag{21}$$

This retains the finite-window top error until its limiting disappearance is justified.

Proposition 8 (Adaptive correlation bound)**.** *Under the hypotheses just stated, with carry averages formed using $U=U_\alpha(-1)$,*

$$\begin{aligned}|\langle g,U_\alpha(t)f\rangle|\le{}&\|g\|\big(\|P_{m,q}f\|+\|P_{m,q+1}f\|\big)\\&+FG\rho(B_m^c)+6FG(\ell+1)w_m.\end{aligned} \tag{22}$$

Proof. Let $g_0=g\mathbf{1}_{\bigcup_{i+v<h}L_i}$ and $g_1=g\mathbf{1}_{\bigcup_{i+v\ge h}L_i}$. They are disjoint masks of g, each of norm at most $\|g\|$. Because the branch depends on i alone, the definition of J_K gives the exact identity

$$J_K=\langle g_0,U_\alpha(t-R(q))P_{m,q,K}f\rangle+\langle g_1,U_\alpha(t-R(q+1))P_{m,q+1,K}f\rangle.$$

Thus $|J_K|$ is at most $\|g\|$ times the sum of the two finite carry norms. As $K\to\infty$, these vectors converge to their infinite carry averages, while $W=w2^{-K}\to 0$. Passing to the limit in (21) proves the stated inequality. Both adjacent queries are essential because the branch changes at the end of a return block. □

6.3. **An integer-window estimate.** For later energy estimates we need the same two-piece coordinate map without requiring its values to stay inside the tower. Suppose $\psi:\mathbb{Z}\to\mathbb{R}$ satisfies $|\psi|\le M$, and use any nonnegative integers c_0,c_1 in (18). Then

$$\left|\sum_{i=0}^{h-1}\psi(z(i))-\sum_{i=0}^{h-1}\psi(i)\right|\le 2M(c_0+\Delta+c_1). \tag{23}$$

Here v, Δ are integers with $0 \le v < h + \Delta$ and $\Delta \ge 0$. To prove it, translating a consecutive integer window to the left by $d \ge 0$ changes its sum by at most $2Md$: cancel the common terms, or telescope d single-step translations. If $v \le h$, the split images are the windows

$$[v - c_0, h - c_0), \qquad [-\Delta - c_1, v - \Delta - c_1).$$

Before those left translations, the corresponding windows $[v, h)$ and $[0, v)$ partition $[0, h)$. Applying the translation bound to each gives (23). If $v > h$, only the second branch occurs, and the entire window is translated left by $d = h + \Delta + c_1 - v$, with $0 \le d \le \Delta + c_1$; the same bound applies. Empty windows cause no exception.

6.4. **From array energy to actual displacement.** Continue with a single observable f, whose label values are bounded by F. Define its zero-extended ordinary array

$$A(z) = \begin{cases} f(z), & 0 \le z < h, \\ 0, & \text{otherwise}, \end{cases} \qquad z \in \mathbb{Z}.$$

The symbol A in this subsection denotes this array; the smoothing operator will be defined separately below. Set

$$Q_K = w\mathbb{E}_r \sum_{i=0}^{h-1} A(i)A(z_r(i)), \qquad E_K = \mathbb{E}_r \sum_{i=0}^{h-1} \big(A(i) - A(z_r(i))\big)^2.$$

Applying (21) with $g = f$ compares the actual correlation with J_K. To compare J_K with Q_K, use the local shift rule on each good level: its integral is exactly $wf(i)f(z_r(i))$. On each bad level the absolute difference is at most $2F^2w$. Hence (17) gives

$$|J_K - Q_K| \le 6F^2(\ell + 1)w.$$

Together these yield

$$|\langle f, U_\alpha(t)f\rangle - Q_K| \le F^2 e + 12F^2(\ell + 1)w + 2F^2(q + 1)hW. \tag{24}$$

For $\psi = A^2$ the integer-window bound and (17) give

$$\mathbb{E}_r \sum_i A(z_r(i))^2 \le \sum_i A(i)^2 + 6F^2(\ell + 1).$$

Also $w\sum_i A(i)^2 = \int_{B_m} f^2\, d\rho \le \|f\|^2$. Expanding the square in E_K, then using (24), therefore proves

$$\begin{aligned} wE_K \le{}& \|U_\alpha(t)f - f\|^2 + 2F^2 e \\ &+ 30F^2(\ell + 1)w + 4F^2(q + 1)hW. \end{aligned} \tag{25}$$

The coefficient 30 is $6 + 2 \cdot 12$; the identity $\|U_\alpha(t)f - f\|^2 = 2\|f\|^2 - 2\langle f, U_\alpha(t)f\rangle$ accounts for the factor two.

The one-step displacement satisfies the companion bound

$$\|U_\alpha(-1)f - f\|^2 \le wD_A + 4F^2(e + w), \qquad D_A = \sum_{i=0}^{h-1} \big(A(i) - A(i-1)\big)^2. \tag{26}$$

Indeed the actual displacement integrand equals the corresponding array term at every ordinary level $i > 0$. Discard the nonnegative array term at $i = 0$, and bound the actual integrand by $4F^2$ on the base and the spacer set, whose measures are w and e.

6.5. **The common marked pair controls one-step energy.** Suppose

$$q_k \neq q_{k+1}, \qquad d_\alpha(m+k+1) = 1,$$
$$(q+1)_{k+j} = q_{k+j} \quad (j = 0, 1, 2).$$

Take $K = k + 3 + L$ for any $L \geq 0$, and interpret a uniform residue as a uniform binary word of length K. Write it as a prefix of length k, three middle bits, and a tail of length L. Put $b = q_k$ and $u = q_{k+2}$. For every fixed prefix and tail, the two middle words

$$(b, 0, u), \qquad (b, 1, u)$$

increase the carry cost by exactly one when passing from the first to the second, for *each* of the queries $q, q+1$. This follows from the reset–branch–reset calculation in Section 5: the first reset removes any dependence on the carry arriving from the prefix, the marked middle digit contributes the difference one, and the last reset makes the outgoing carries equal so the tail contributions agree. The common three query bits let us use the same two input words for both queries.

Consequently, for either branch of (18), the coordinate for the second word equals the coordinate for the first minus one. Define

$$\psi(z) = (A(z) - A(z-1))^2, \qquad 0 \leq \psi(z) \leq 4F^2,$$

and let D be the average, over the prefix and tail, of $\sum_i \psi(z_-(i))$, where z_- is the coordinate belonging to the selected middle word $(b, 0, u)$. For each level and context the elementary inequality

$$|x - y|^2 \leq 2|x - z|^2 + 2|y - z|^2$$

applied with $x = A(z_-(i))$, $y = A(z_-(i) - 1)$, $z = A(i)$ gives a lower bound on the sum of the two selected displacement energies. Each of the eight middle words has conditional probability $1/8$; all other energies are nonnegative. Summing and averaging therefore gives

$$D \leq 16E_K. \tag{27}$$

No independence between the carry state and the prefix is used.

The nonnegative cost $\gamma = c_0 + \Delta + c_1$ has full-word mean at most $3(\ell+1)$. Its conditional mean with the selected middle word fixed is thus at most $8 \cdot 3(\ell+1)$, because that word has probability $1/8$. Apply (23) to ψ, whose bound is $4F^2$, and then average over the prefix and tail. It follows that

$$D_A \leq D + 8F^2\mathbb{E}(\gamma \mid \text{middle} = (b, 0, u)) \leq 16E_K + 192F^2(\ell+1). \tag{28}$$

This derives the paired-window constant, including the cost of conditioning on a selected word.

Proposition 9 (Marked displacement estimate)**.** *Under the bracket, size, and common-mark hypotheses above,*

$$\|U_\alpha(-1)f - f\|^2 \leq 16\|U_\alpha(t)f - f\|^2 + 36F^2\rho(B_m^c) + 676F^2(\ell+1)w_m. \tag{29}$$

Proof. Insert (28) into (26), then use (25). The spacer coefficient is $16 \cdot 2 + 4 = 36$. The width coefficient before its final simplification is

$$16 \cdot 30(\ell+1) + 192(\ell+1) + 4 = 672(\ell+1) + 4 \leq 676(\ell+1).$$

The finite-window remainder is exactly $64F^2(q+1)hW$, where $W = w_{m+k+3+L}$. This proves the finite estimate with that extra term. Now hold m, q, k, ℓ, t fixed and let $L \to \infty$. Exact refinement gives $W = w2^{-(k+3+L)} \to 0$, while every other term is independent of L. This proves (29) with all three numerical constants derived. □

7. Rigidity, mixing, and the use of irrationality

7.1. **A complete zero-block displacement estimate.** We now derive the quantitative estimate from consecutive identical copies, including its square-integral boundary term. Fix a stage n and an integer $L \geq 0$ such that

$$d_\alpha(n+j) = 0 \qquad \text{whenever } j+1 < L. \tag{30}$$

Put $N = 2^L$ and $h = h_n$. For $0 \leq r < N$, the return formula (4) gives

$$R_n(r) = rh. \tag{31}$$

To check the possible last term, its quotient is $\lfloor r/2^L \rfloor = 0$; every earlier term has zero digit by (30). Thus no condition is needed on the final digit $d_\alpha(n+L-1)$. Also $Nh \leq h_{n+L}$ by the height recurrence.

Let $f = a(\lambda_n(0))$ be a stage-n observable with $|a(i)| \leq F$ on all labels, including the spacer label. Let S be the prefix consisting of the first $(N-1)h$ ordinary levels of the stage-$(n+L)$ tower. On S, translation by h moves each stage-n ordinary level to the same level in the next copy. Indeed (31) shows that the copies start at $0, h, \ldots, (N-1)h$, and both the starting and destination levels stay below $Nh \leq h_{n+L}$. The local translation rule and refinement therefore imply

$$f \circ \sigma^h = f \quad \text{almost everywhere on } S. \tag{32}$$

For $L = 0$ the set S is empty; the formulas below still apply.

Write

$$A = \sum_{i=0}^{h-1} a(i)^2, \quad w = w_n, \quad W = w_{n+L}, \quad e = \rho(B_n^c), \quad E = \|f\|_2^2.$$

Equal-level measure and the single spacer label give

$$E = wA + e\,a(h)^2.$$

There are $N-1$ complete copies inside S, each with level mass W. By (15), $w = NW$, so

$$\int_S f^2\,d\rho = (N-1)WA,$$
$$\int_{S^c} f^2\,d\rho = WA + e\,a(h)^2 \leq E/N + F^2 e.$$

This is an estimate for the *square-integral mass* of the exceptional set, not only its measure. It retains $\|f\|^2$ rather than replacing the whole function by its uniform bound.

Set $g = f \circ \sigma^h$. Invariance of ρ gives $\int g^2 = \int f^2$, and (32) gives equality of their square integrals on S. Their square integrals on S^c consequently coincide as well. Since $(g-f)^2 \leq 2g^2 + 2f^2$ and the displacement vanishes on S,

$$\|g - f\|_2^2 \leq 4 \int_{S^c} f^2\,d\rho \leq 4\,2^{-L}\|f\|_2^2 + 4F^2\rho(B_n^c).$$

For an earlier-stage observable f at stage $k \leq n$, compatible refinement represents it at stage n without increasing F. We have therefore proved, with exactly the source's hypotheses,

$$\|U_\alpha(h_\alpha(n))f - f\|^2 \leq 4\,2^{-L}\|f\|^2 + 4F^2\rho(B_n^c). \tag{33}$$

This closes the zero-block estimate directly from return arithmetic and the finite-stage measure relations proved above.

With (10) and density, it has two precise consequences:

(1) If $n_r, L_r \to \infty$ and the indicated zero blocks occur, then $U_\alpha(h_\alpha(n_r))f \to f$ for every $f \in L^2(\rho)$.

(2) If merely $d_\alpha(n_r) = 0$ and $n_r \to \infty$, use $L = 2$ to obtain, for every f and $\varepsilon > 0$, eventually

$$\|U_\alpha(h_\alpha(n_r))f - f\| \le \|f\| + \varepsilon.$$

Both statements also hold at the negative times, since the displacement norm of an isometry equals that of its inverse.

For cross heights, take $t_n = h_\beta(n)$, $m = \lfloor n/2 \rfloor$, and $L = n - m$. The bracket query $q = q_m(t_n)$ obeys

$$\frac{q}{2^L} \longrightarrow \frac{\beta}{\alpha}, \qquad \frac{q+1}{2^L} \longrightarrow \frac{\beta}{\alpha}. \tag{34}$$

Indeed (6) gives $t_n/(h_m + 1) - 1 < q \le t_n/h_m$, and division by 2^L, followed by $h_m/2^m \to \alpha$ and $t_n/2^n \to \beta$, proves the claim. Also $q + 1 \le 2^{n+C}$ for some constant C depending only on the two parameters. Thus choosing $\ell = n + C$ makes the error width

$$(\ell + 1)w_m \le (n + C + 1)2^{-\lfloor n/2 \rfloor} \longrightarrow 0.$$

If a real number c is irrational, every fixed finite binary prefix is stable under sufficiently small perturbations of c: after multiplication by a fixed power of two, c lies strictly between adjacent integers. Apply this to (34). A transition at the digit indices $b, b + 1$ of $c = \beta/\alpha$ appears in both query words at bit positions $L - b - 2, L - b - 1$. Their three bits around that transition eventually agree as well. Nondyadicity supplies arbitrarily many separated transitions by Lemma 3.

7.2. **Amplification of disjoint marked blocks.** We give the complete block argument. A block B consists of query bits and digit weights of a common finite length b. For an input word $x \in \{0, 1\}^b$ and incoming carry $c \in \{0, 1\}$, write $e_B(x, c)$ for its outgoing carry and $v_B(x, c)$ for its weighted carry cost. Define

$$T_{B,c}f = 2^{-b} \sum_x U^{v_B(x,c)} f.$$

This is a contraction and commutes with U and hence with $A = (I + U)/2$. If D is the following block, independence of the uniform input words and addition of the costs give the exact composition identity

$$T_{BD,c}f = 2^{-b} \sum_x U^{v_B(x,c)} T_{D,e_B(x,c)} f. \tag{35}$$

The outgoing carry is retained in the suffix operator; it cannot be averaged away before composition.

Call B marked if, for each incoming c, there are two distinct input words x_0, x_1 with the same outgoing state e and costs $v, v + 1$. The two selected terms in (35) then sum to

$$U^v \big(T_{D,e}f + UT_{D,e}f\big) = 2U^v T_{D,e}Af.$$

Their total probability is $p = 2 \cdot 2^{-b} \le 1$. If, uniformly in the incoming suffix state,

$$\|T_{D,e}f\| \le M, \qquad \|T_{D,e}Af\| \le Q,$$

separating those two words from all other words and using the triangle inequality proves

$$\|T_{BD,c}f\| \le (1 - p)M + pQ. \tag{36}$$

The selected words may depend on c; the bound remains uniform because M, Q do.

Proposition 10 (Quantitative amplification)**.** *Suppose a finite query/digit word contains N disjoint marked blocks in order, separated by arbitrary unmarked words. Suppose each selected pair has probability at least δ, where $0 \le \delta \le 1$. If*

$$\|A^j f\| \le \varepsilon + Cs^j \quad (j \ge 0), \qquad C \ge 0, \quad 0 \le s \le 1,$$

then its carry average, for either incoming carry, has norm at most

$$\varepsilon + C(1 - \delta + \delta s)^N.$$

The same bound holds for the infinite carry average if these marked blocks lie in a finite initial window.

Proof. Put $\rho_* = 1 - \delta + \delta s \geq 0$. Induct on the block decomposition, allowing both the vector and its profile constant to vary. With no marked blocks, contractivity gives the bound $\|f\| \leq \varepsilon + C$. An unmarked prefix changes no bound: apply (35), bound each suffix average uniformly, and use that powers of U are isometries.

For a marked first block followed by a suffix with N marked blocks, the induction hypothesis gives

$$M = \varepsilon + C\rho_*^N.$$

The vector Af has profile

$$\|A^j Af\| \leq \varepsilon + (Cs)s^j,$$

so the same hypothesis gives $Q = \varepsilon + Cs\rho_*^N$. By (36),

$$\begin{aligned}\|T_{BD,c}f\| &\leq (1-p)(\varepsilon + C\rho_*^N) + p(\varepsilon + Cs\rho_*^N) \\ &= \varepsilon + C(1 - p + ps)\rho_*^N \leq \varepsilon + C\rho_*^{N+1},\end{aligned}$$

since $p \geq \delta$ and $1 - s \geq 0$. This proves the finite result, including arbitrary unmarked tails. Extend the window by any number of higher bits; the same marked blocks persist and the extension is an unmarked tail. Thus the bound is uniform in the extended length. The norm convergence of finite carry averages proved in Section 5 passes it to the infinite average. □

In particular the previously needed estimate is now proved:

$$\|P_{m,q}f\| \leq \varepsilon + C(1 - \delta + \delta s)^N. \tag{37}$$

7.3. **Elementary smoothing and decay.** The smoothing convergence used with this proposition does not require a spectral representation. For every vector g, the parallelogram identity gives

$$\|Ag\|^2 + \tfrac{1}{4}\|g - Ug\|^2 = \|g\|^2. \tag{38}$$

Since A is a contraction commuting with U, the sequence $\|A^j(g - Ug)\|$ is nonincreasing. Summing (38) with g replaced by $A^j g$, for $0 \leq j \leq n$, therefore gives

$$(n+1)\|A^n(g - Ug)\|^2 \leq \sum_{j=0}^{n} \|A^j(g - Ug)\|^2 = 4\big(\|g\|^2 - \|A^{n+1}g\|^2\big) \leq 4\|g\|^2.$$

Consequently $A^n f \to 0$ for every coboundary $f = g - Ug$.

The orthogonal complement of the range of $I - U$ is precisely the fixed-vector space. Indeed h is orthogonal to every $g - Ug$ if and only if $(I - U^{-1})h = 0$, equivalently $Uh = h$. Taking orthogonal complements again shows that every vector orthogonal to the fixed space lies in the closure of the coboundaries. Uniform contractivity of A^n transfers convergence from that range to its closure: if $\|f - (g - Ug)\| < \eta$, then

$$\limsup_n \|A^n f\| \leq \eta.$$

Thus $A^n f \to 0$ for every such f. In our tower system the fixed space consists of constants by Proposition 7, so this applies to every mean-zero vector.

For $0 < s < 1$ and $\varepsilon > 0$, this convergence yields the precise profile needed above. Choose K so $\|A^j f\| \leq \varepsilon$ for $j \geq K$, and set $C = \|f\|\, s^{-K}$. For $j < K$, $Cs^j \geq \|f\| \geq \|A^j f\|$; hence

$$\|A^j f\| \leq \varepsilon + Cs^j \qquad (j \geq 0).$$

Fix $s = 1/2$. If a sequence of query/digit words eventually contains N marked blocks of weight at least a fixed $\delta > 0$, for each fixed N, then its infinite carry averages tend to zero on f. To check the quantifiers, given $\eta > 0$ choose the profile with $\varepsilon = \eta/2$, next choose N with $C(1 - \delta/2)^N < \eta/2$,

and finally take the query index large enough to supply those N blocks. Equation (37) then bounds the norm by η.

7.4. **Bounded zero runs supply the marked blocks.** We expand the delayed-path construction to verify the hypotheses just used. Suppose a query transition starts at bit k, with query bits $a, 1-a$, and choose the first digit one at position $m+k+1+r$. Thus

$$d_\alpha(m+k+1+j) = 0 \ (0 \le j < r), \qquad d_\alpha(m+k+1+r) = 1.$$

Use the block from k through $k+r+2$. Write its query word as

$$(a, 1-a, q_2, \dots, q_{r+1}, e).$$

For each $c \in \{0,1\}$ choose the input word

$$(a, c, 1-q_2, \dots, 1-q_{r+1}, e).$$

The first bit resets any incoming carry to a; the second outputs c; the middle complementary bits preserve c; the last resets it to e. Hence its weighted cost is

$$d_\alpha(m+k)a + c\sum_{j=0}^{r} d_\alpha(m+k+1+j) + d_\alpha(m+k+r+2)e.$$

The sum equals one. Taking $c = 0, 1$ therefore gives distinct words with the same outgoing carry and costs differing by exactly one, for either incoming carry. This proves markedness, including $r = 0$.

If every H-digit window contains a one, its first occurrence has $r < H$, so the block length $r+3$ is at most $H+2$. Its selected probability is at least

$$\delta = 2 \cdot 2^{-(H+2)} > 0.$$

Query transitions separated by at least $H+2$ thus give disjoint marked blocks, provided their start positions k satisfy $k + H + 2 \le L$ in the chosen window $[0, L)$.

For completeness these margins can be met for the cross-height queries. Given N, choose N transition indices b_i of the irrational ratio β/α with $b_i \ge H$ and pairwise separation at least $H+2$; unbounded transitions allow a greedy choice. For large n, both scaled queries in (34) have those transitions at $k_i = L - b_i - 2 \ge 0$. Also

$$k_i + H + 2 = L - b_i + H \le L.$$

Order these positions increasingly, reversing the order of the b_i. Their separation is unchanged, so the delayed blocks are disjoint and entirely inside the window. Stability of the finitely many binary digits holds simultaneously for q and $q+1$. This establishes, for each fixed N, the eventual marked-block count for both queries with the same δ.

7.5. **Bounded zero runs imply cross-height mixing.** The amplification and smoothing proofs now give $P_{m,q} f \to 0$ and $P_{m,q+1} f \to 0$ for mean-zero f. Proposition 8, the body bounds, and the vanishing width then imply

$$\langle g, U_\alpha(h_\beta(n)) f \rangle \longrightarrow 0 \qquad (\textstyle\int f \, d\rho = 0). \tag{39}$$

Initially this holds for tower observables, after refining them to stage m; density and the isometry norm extend it to arbitrary L^2 vectors.

7.6. **A marked spacer excludes nonconstant cross-height rigidity.** Fix a transition $b > 0$ of the binary digits of β/α. Suppose $a_r \to \infty$, $d_\alpha(a_r) = 1$, and $n_r = a_r + b + 1$. At the half-stage $m_r = \lfloor n_r/2 \rfloor$, set

$$k_r = n_r - m_r - b - 2.$$

Then $m_r + k_r + 1 = a_r$. For sufficiently large r, the three query bits agree for q and $q+1$ and are marked at k_r, by (34). Proposition 9 applies, with errors tending to zero. If $U_\alpha(h_\beta(n_r))f \to f$, it follows that $U_\alpha(-1)f = f$. The density step is explicit: for a bounded tower observable g close to f,

$$\limsup_r \|U_\alpha(h_\beta(n_r))g - g\| \le 2\|g - f\|.$$

Taking square roots in (29) and letting the errors vanish gives $\|U_\alpha(-1)g - g\| \le 8\|g - f\|$. Therefore

$$\|U_\alpha(-1)f - f\| \le 10\|g - f\|.$$

Let $g \to f$. Ergodicity makes f constant; if it has mean zero, it vanishes.

8. The Hilbert-space joining argument

For an anti-joining η between (X, μ, σ) and (X, ν, σ), define its contraction $J : L^2(\nu) \to L^2(\mu)$ by

$$\langle g, Jf \rangle = \int g(x) f(y)\, d\eta(x, y).$$

Cauchy–Schwarz and the marginals show $\|J\| \le 1$, $J1 = 1$, and preservation of means. Anti-invariance gives

$$U_\mu(t)J = JU_\nu(-t).$$

If J kills every mean-zero vector, then $Jf = (\int f\, d\nu)1$, so applying the formula to indicator functions yields product measure on every measurable rectangle, hence $\eta = \mu \times \nu$.

To use the labeled estimates, one lifts each symbolic name limit to a labeled-name limit along the same empirical subsequence. Compactness and continuity of the projection provide this lift. The pullbacks are isometries e, d between the respective L^2 spaces. The lifted operator eJd^* preserves constants and means and intertwines the labeled actions; restriction back to the factors recovers J. The source carries out this lifting explicitly. It is a functional-analytic construction, not an assumption that every joining has an arbitrarily chosen invariant lift.

Lemma 11 (Mixing against partial rigidity)**.** *Let T_r, S_r be isometries on real Hilbert spaces with constant vectors, and let J preserve constants and means and satisfy $T_r J = J S_r$. Suppose T_r converges weakly to zero on mean-zero vectors and*

$$\limsup_r \|S_r v - v\| \le \|v\| \quad \textit{for every } v.$$

Then J vanishes on the mean-zero subspace.

Proof. For a mean-zero vector g in the target space let $v = J^* g$, which also has mean zero. Then

$$\langle v, S_r v \rangle = \langle g, T_r J J^* g \rangle \longrightarrow 0.$$

On the other hand, the isometry identity gives

$$\|S_r v - v\|^2 = 2\|v\|^2 - 2\langle v, S_r v \rangle.$$

The assumed bound forces $2\|v\|^2 \le \|v\|^2$, hence $v = 0$. Thus J^* kills all mean-zero target vectors. For mean-zero f, Jf is mean zero and $\|Jf\|^2 = \langle J^* J f, f \rangle = 0$. □

Apply this with $T_r = U_\alpha(h_\beta(n_r))$ and $S_r = U_\beta(-h_\beta(n_r))$, where $n_r \to \infty$ are zero-digit indices of β. Such indices always exist. Cross-height mixing (39) supplies the first hypothesis; the single-zero partial rigidity consequence of (33) supplies the second. Hence

$$\text{bounded zero runs in } d_\alpha \quad \Longrightarrow \quad D(\alpha, \beta) \tag{40}$$

when the ratio is irrational.

A second operator observation is simpler. If $S_r f \to f$ for all f, while the only mean-zero vector g with $T_r g \to g$ is zero, then $Jf = 0$ for every mean-zero f: intertwining and continuity give $T_r Jf = JS_r f \to Jf$. Thus long zero blocks for the β system, combined with marked rigidity exclusion for the α system, imply $D(\alpha, \beta)$.

9. Digit transport closes the argument

Let $A, B \subset \mathbb{N}$. Say there is a *forward transport with shift* b from A to B if there are integers $L > 0, N$ such that every $a \in A$, $a \geq N$, has some

$$y \in B \cap [a + b, a + b + L).$$

These transports compose: shifts add and widths add. They preserve unboundedness. A strictly positive self-transport on an unbounded set implies bounded gaps: start with one occurrence above N and iteratively choose its forward successor. This produces an unbounded increasing sequence of occurrences with consecutive gaps below $b+L$; enlarge the bound to cover the finite initial segment.

Put $A_\alpha = \{n : d_\alpha(n) = 1\}$. Suppose the ratio is irrational and $D(\alpha, \beta)$ fails. Choose a transition index $b > 0$ in the digits of β/α. We claim that A_α transports to A_β with shift $b + 1$. If not, negating the quantifiers in the transport definition gives, for every r, an $a_r \geq r$ in A_α such that

$$d_\beta(a_r + b + 1 + j) = 0 \qquad (0 \leq j < r + 2).$$

Set $n_r = a_r + b + 1$. These are growing zero blocks for the β system, so its operators at times $-h_\beta(n_r)$ converge strongly to the identity. The marked spacer at a_r, aligned with the fixed transition at b, excludes nonconstant rigidity in the α system at the positive cross heights. The second operator observation therefore gives $D(\alpha, \beta)$, a contradiction. We have established

$$\neg D(\alpha, \beta) \Longrightarrow \text{a positive-shift transport } A_\alpha \longrightarrow A_\beta. \tag{41}$$

We now prove the final deduction from (40) and (41). At least one of α, β is irrational, since otherwise their ratio is rational. Exchange the parameters if necessary so that A_α is unbounded, by Lemma 3. If D failed, (41) would give a positive transport $A_\alpha \to A_\beta$, and in particular A_β would be unbounded. Symmetry of D and irrationality of the inverse ratio give a second positive transport $A_\beta \to A_\alpha$. Their composition is a positive self-transport of A_α, so d_α has bounded zero runs. But (40) then gives D, the contradiction. Thus D holds. Proposition 5 gives completeness for normalized parameters, and the initial tail scaling gives Theorem 1.

This completes the proof, using the correlation, displacement, and carry-amplification estimates established above.

Appendix A. Source identity and declaration map

The formal solution credited to JenW1N is available at

`https://conjectures.io/results/815c1d5f-3afb-4430-8e2c-260d9038f5b0/solution`.

The supplied file has 10,152 lines. Its SHA-256 is

`7188a481c218751aa666a8577f0e6b1281462d4d3d97d548a565db99b44cc2ac`.

The final declaration is `target` at line 10148, with target type `Erdos354.erdos_354.parts.i`. It instantiates `full_target_of_symbolic_digit_criteria` with two proved criteria. Earlier declarations named `full_target_of_dynamical_criteria` and `full_target_of_normalized` are explicitly conditional reductions and must not be mistaken for the completed result in isolation.

Component	Declarations and lines
Transport arithmetic	`ForwardTransport.compose`, `boundedZeroRuns_of_forward_return`, `boundedZeroRuns_of_two_transports`, 30–73.
Floor digits	`digit_zero_or_one`, `height_sum_gap`, `unboundedZeros`, `unboundedTransitions`, 84–248.
Normalization and target	`CompletePair`, `complete_of_dyadic_scale`, `full_target_of_normalized`, 264–360.
Finite sums	`finiteSums_disjoint_shift`, `card_finiteSums`, 427–507.
Missing-sum obstruction	`incompletePair_nonproduct_joining`, `SymbolicallyDisjoint`, `completePair_of_symbolicallyDisjoint`, 917–1109.
Carry paths	`marked_three_bit_paths`, `returnPosition_add`, `returnPosition_eq_binary_sum`, 1141–1455.
Tower construction	`fullReturnPosition`, `towerLabel`, `towerProjection_name`, 1506–1701.
Exact tower refinement	`fullReturnPosition_pow_mul_add`, `towerLabel_refines`, `towerLabel_add_eq`, 1779–1834, 2077–2175.
Level copies and shifts	`collapseLevels_eq_copyOffset_iff`, `ae_level_copies`, `ae_shift_level`, 3190–3227, 3860–3880.
Density and ergodicity	Finite-window approximation and conditional-expectation proofs above; formal counterparts 2722–3095, 4874–5234, 6007–6107, `fixed_vector_eq_const`, 7886–7899.
Zero-block displacement	`fullReturnPosition_of_zero_digits`, `square_displacement_le_compl`, `zero_digits_displacement_integral`, 4600–4611, 4728–4774, 4795–4870.
Mean carry cost	`carryCost_mean_le`, 1976–2045.
Tower body bounds	`level_measure`, `body_real`, `outside_mass_bound`, 2336–2543.
Marked contraction	`markedCarryAverage_bound`, `infiniteCarryAverage_marked_bound`, 5346–5863.
Two-piece cells and counts	`badCarryLevels_card_le`, `setIntegral_adaptive_copy`, `carryCost_with_top_sum_le`, 4097–4130, 4210–4228, 4426–4437.
Finite correlation comparison	`finite_adaptive_correlation_error`, `adaptive_finite_inner`, 4561–4590, 6320–6332.
Carry-to-correlation	`correlation_bound_by_carries`, `correlation_tendsto_of_carry_decay`, 6359–6544.
Selected-pair composition	`carryWordAverage_append`, `carryWordAverage_selected_pair`, 6774–6810.
Elementary smoothing	`halfUnitaryAverage_energy_sum`, `halfUnitaryAverage_coboundary_bound`, 6676–6719; `halfUnitaryAverage_tendsto_zero_of_orthogonal_fixed`, 6898–6908.
Finite amplification	`carryWordAverage_geometric_bound`, 6938–6987.
Delayed pairs and counting	`delayed_marked_pair`, `hasCarryPair_window_first_one`, `hasCarryPairCount_of_separated_transitions`, 7047–7060, 7238–7251, 7285–7339.
Block amplification	`infiniteCarryAverage_geometric_bound`, `infiniteCarryAverage_tendsto_zero_of_eventual_pairs`, 7148–7187.
Irrational queries	`eventually_carry_pairs_of_irrational_limit`, 7438–7465.

Component	Declarations and lines
Zero-block rigidity	`zero_blocks_rigid`, `zero_digits_partial_rigidity`, 7506–7688.
Joining operators	`antiJoiningOperator`, `towerJoiningOperator`, `symbolicallyDisjoint_of_tower_intertwiners`, 8449–8750.
Operator criteria	`intertwiner_vanishes_of_mixing_partial_rigidity`, `intertwiner_vanishes_of_rigidity_exclusion`, 8768–8806.
Bounded-zero disjointness	`boundedZeros_height_mixing`, `symbolicallyDisjoint_of_boundedZeroRuns`, 8907–8926.
Integer windows and pair energy	`split_window_sum_bound`, `split_word_pair_energy`, 9133–9206, 9393–9466.
Common pair and array comparison	`wordCarryCost_common_pair`, `array_correlation_error`, `array_displacement_energy_bound`, 9510–9530, 9648–9667, 9742–9779.
Marked displacement	`finite_marked_displacement_bound`, `marked_displacement_bound`, 9781–9836.
Rigidity exclusion	`eventually_marked_cross_queries`, `cross_marked_rigidity_exclusion`, 9953–10058.
Final transport and target	`not_digitTransport_zero_blocks`, `forwardTransport_of_not_symbolicallyDisjoint`, `target`, 10099–10151.

Appendix B. Reconstruction and review record

The manuscript now supplies proofs of the finite-stage refinement and weak-limit transfer, tower density and ergodicity, zero-block displacement, marked-block amplification and smoothing, separated-block counting, adaptive correlation, and marked displacement. No analytical proposition is intentionally left as an unproved dependency on the formal source.

The correlation argument first proves the finite error

$$FG\rho(B_m^c) + 6FG(\ell+1)w_m + 2FG(q+1)h_m w_{m+K},$$

then lets $K \to \infty$. For displacement, the array energy contributes a factor of 30, the selected pair contributes 192, and the boundary contributes 4. These estimates give the final constants 16, 36, and 676. The finite remainder

$$64F^2(q+1)h_m w_{m+k+3+L}$$

is kept until $L \to \infty$. The source map above records the corresponding formal components.

This is a mathematical reconstruction, not a claim that the prose is kernel-checked. The supplied Lean text depends on its original imports and theorem environment, and that environment has not been freshly built for this manuscript. A separate agent reviewed the expanded analytical proofs, tower structure, rigidity consequences, and final operator and transport arguments and found no substantive gap. That review also checked the carry signs and quantitative constants against the formal declarations. No external peer review is claimed. Successful TeX compilation and page inspection establish document integrity only; mathematical reliability also depends on the stated hypotheses and proof arguments.