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ERDŐS PROBLEM 96: SUPERLINEARLY MANY UNIT DISTANCES IN CONVEX POSITION

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ABSTRACT. For every integer $d \geq 2$, we construct a finite set in the Euclidean plane, in strictly convex position, with $N_d = 2d^d 2^{d^{d+1}}$ vertices and at least $(d/4)N_d$ unordered pairs at distance one. Thus the maximum number of unit distances among the vertices of a convex n-gon is not $O(n)$, giving a negative answer to Erdős Problem 96. The construction starts from a finite coordinate-incidence graph encoded by powers of five. Two families of planar seed points have strict support margins and incident distances with fourth-order errors. Rotations of third order correct these errors. A simultaneous perturbation of the seed radii separates all subset products of the rotations while preserving the support margins and exact unit distances. Taking every subset-product copy amplifies the incidence graph into the required convex configuration. The argument is accompanied by a Lean proof; we give its correspondence with the exposition and its verification records.

1. Introduction and main result

For a finite set $V \subset \mathbb{R}^2$, let

$$u(V) = \left|\{\{x, y\} \subseteq V : x \neq y,\ |x - y| = 1\}\right|.$$

Distances are Euclidean, and each unordered pair is counted once. We say that V is in *strictly convex position* if $x \notin \operatorname{conv}(V \setminus \{x\})$ for every $x \in V$. For $|V| \geq 3$, these sets are precisely the vertex sets of strictly convex polygons. Write

$$U_c(n) = \max\{u(V) : |V| = n,\ V \text{ is in strictly convex position}\}.$$

The maximum exists: the admissible family is nonempty and its possible counts are integers between zero and $\binom{n}{2}$. Erdős Problem 96 [1] asks whether $U_c(n) = O(n)$.

Theorem 1.1. *For every integer $d \geq 2$, there is a set $V_d \subset \mathbb{R}^2$ in strictly convex position such that*

$$|V_d| = N_d := 2d^d 2^{d^{d+1}}, \qquad u(V_d) \geq \frac{d}{4} N_d. \tag{1}$$

Corollary 1.2. *For every real C and every integer $N_0 \geq 0$, there is a finite convex vertex set $V \subset \mathbb{R}^2$ satisfying $|V| \geq N_0$ and $u(V) > C|V|$. In particular, $U_c(n)$ is not $O(n)$.*

Proof. Choose an integer $d \geq 2$ with $d > 4C$ and $d \geq N_0$. Since $N_d \geq d$, Theorem 1.1 supplies the required set. This contradicts any proposed eventual linear upper bound for U_c. ☐

The construction gives the quantitative bound in (1) along the sequence (N_d). It does not assert that this is the optimal growth rate of $U_c(n)$.

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1.1. **Proof strategy.** We use the complex plane as an isometric copy of $\mathbb{R}^2$. The starting incidence graph has two vertex classes, each of size d^d, and $m = d^{d+1}$ edges. We assign a nonzero complex seed b_i to each vertex and a unit complex number w_e to each edge. An incident pair $(l(e), p(e))$ satisfies

$$|b_{l(e)} - b_{p(e)} w_e| = 1.$$

For every subset $S \subseteq E$, multiply every seed by $W_S = \prod_{e \in S} w_e$. The prospective vertices are $b_i W_S$. Every edge e then creates a unit pair between the copies with labels $(l(e), S)$ and $(p(e), S \cup \{e\})$, for each S not containing e. This gives $m 2^{m-1}$ pairs on $2d^d 2^m$ vertices.

Two geometric issues must be resolved before this count is valid: the vertices must be distinct, and they must all remain extreme points. Distinct seed radii and distinct products W_S give the first property. Strict radial support margins, larger than the total rotational error, give the second. Sections 3–6 construct seeds and rotations meeting both requirements.

1.2. **Relation to the earlier linear-bound claim.** Khopkar's preprint [2], revised in April 2017, claims a linear upper bound for unit-distance graphs on convexly independent planar point sets. Theorem 1.1 contradicts that asserted bound. Our proof proceeds by a separate construction. We do not locate a specific erroneous step in the earlier argument here.

2. Amplification by subset products

Identify $\mathbb{C}$ with the real inner-product plane, and define the radial support quantity

$$\sigma(a, b) = \operatorname{Re}\big(\overline{a}(b - a)\big) = \langle a, b - a\rangle. \tag{2}$$

Lemma 2.1 (Strict radial support)**.** *Let $(z_i)_{i \in I}$ be a finite family of distinct complex numbers. If $\sigma(z_i, z_j) < 0$ whenever $i \neq j$, then its image is in strictly convex position.*

Proof. For fixed i, every other point lies in the open half-plane $\{z : \langle z_i, z\rangle < |z_i|^2\}$. This half-plane is convex and does not contain z_i, so the convex hull of the other points cannot contain z_i. □

Lemma 2.2 (Rotational error)**.** *Let E be finite and $|w_e| = 1$ for all $e \in E$. Put*

$$H = \sum_{e \in E} |w_e - 1|, \qquad W_S = \prod_{e \in S} w_e.$$

Then $|W_S| = 1$ and $|W_S - 1| \le H$ for every $S \subseteq E$. If $|a|, |b| \le 1$ and $|u| = |v| = 1$ with $|u - 1|, |v - 1| \le H$, then

$$\sigma(au, bv) \le \sigma(a, b) + 2H. \tag{3}$$

Proof. The norm identity is multiplicativity. Repeatedly use $|uv - 1| \le |u - 1| + |v - 1|$ for unit complex numbers to bound the subset products. Also

$$\sigma(au, bv) - \sigma(a, b) = \operatorname{Re}\big(\overline{a} b(\overline{u} v - 1)\big) \le |a||b|\big(|u - 1| + |v - 1|\big) \le 2H.$$

□

Proposition 2.3 (Amplification)**.** *Let L and P be disjoint finite label sets, and let E be a finite nonempty edge set with endpoint maps $l : E \to L$ and $p : E \to P$. Suppose $e \mapsto (l(e), p(e))$ is injective. Put $I = L \sqcup P$. Assume that seeds $b_i \in \mathbb{C}$ and rotations $w_e \in \mathbb{C}$ satisfy:*

1. *$0 < |b_i| < 1$, and all the numbers $|b_i|$ are distinct;*
2. *$|w_e| = 1$, and the products W_S, $S \subseteq E$, are distinct;*
3. *$|b_{l(e)} - b_{p(e)} w_e| = 1$ for every $e \in E$;*
4. *$\sigma(b_i, b_j) < -2 \sum_{e \in E} |w_e - 1|$ whenever $i \neq j$.*

Then

$$V = \{b_i W_S : i \in I,\ S \subseteq E\}$$

is in strictly convex position, with

$$|V| = (|L| + |P|)2^{|E|}, \qquad u(V) \geq |E|2^{|E|-1}. \tag{4}$$

Proof. If $b_i W_S = b_j W_T$, taking absolute values gives $|b_i| = |b_j|$, hence $i = j$. Since $b_i \neq 0$, cancellation and distinctness of the products give $S = T$. This proves the vertex count.

For vertices from different seeds, Lemma 2.2 and condition (4) give negative support. For two distinct vertices x, y from the same seed, $|x| = |y|$ and

$$2\sigma(x, y) = |y|^2 - |x|^2 - |y - x|^2 = -|y - x|^2 < 0.$$

Lemma 2.1 proves strict convex position.

For each $e \in E$ and $S \subseteq E \setminus \{e\}$, the pair

$$\{b_{l(e)} W_S,\ b_{p(e)} W_{S \cup \{e\}}\} \tag{5}$$

has distance one, because its difference is $(b_{l(e)} - b_{p(e)} w_e) W_S$. These unordered pairs are distinct. Indeed, the injective vertex labeling recovers the L endpoint and its subset S, as well as the P endpoint; their order cannot be reversed because L and P are disjoint. The endpoint maps then recover e. There are $2^{|E|-1}$ choices of S for each e, proving the lower bound. Additional unit pairs, if present, only increase $u(V)$. □

3. A coordinate-incidence pattern and its digits

Fix an integer $d \geq 2$ and write $D = \{0, \ldots, d-1\}$. Let $P = D^D$ be the set of functions $f : D \to D$. Let L consist of pairs (i, g), where $i \in D$ and $g : D \setminus \{i\} \to D$. Set $E = P \times D$, and assign to $e = (f, i)$ the endpoints

$$p(e) = f, \qquad l(e) = (i, f|_{D \setminus \{i\}}).$$

Both endpoints together determine e, and

$$|P| = d^d, \qquad |L| = d\, d^{d-1} = d^d, \qquad |E| = d^{d+1}. \tag{6}$$

Choose an injective numbering $\kappa : D \times D \to \mathbb{N}$; for example, $\kappa(i, j) = di + j$. Associate digit sets

$$A_f = \{\kappa(j, f(j)) : j \in D\}, \qquad A_{(i,g)} = \{\kappa(j, g(j)) : j \in D \setminus \{i\}\}.$$

Within each label class these sets are distinct. A point digit set recovers every coordinate value. A line digit set recovers the missing coordinate and all remaining values. For $e = (f, i)$,

$$A_{p(e)} = A_{l(e)} \cup \{\kappa(i, f(i))\}, \qquad \kappa(i, f(i)) \notin A_{l(e)}. \tag{7}$$

For a finite set $A \subset \mathbb{N}$, define

$$X(A) = \sum_{j \in A} 5^j, \qquad T(A) = 1 + \sum_{j \in A} 25^j.$$

Lemma 3.1 (Digit separation)**.** *If $A \neq B$, then $X(A) \neq X(B)$, $T(A) \neq T(B)$, and*

$$|T(B) - T(A)| < \frac{50}{27}\big(X(B) - X(A)\big)^2. \tag{8}$$

Proof. Cancel common digits and let h be the largest remaining digit. Exchange A, B if necessary so that $h \in B \setminus A$. For $q \in \{5, 25\}$, the difference of the digit sums lies between $q^h - \sum_{j<h} q^j$ and $q^h + \sum_{j<h} q^j$. In particular both differences are positive. The geometric sums give

$$X(B) - X(A) > \frac{3}{4}\, 5^h, \qquad 0 < T(B) - T(A) < \frac{25}{24}\, 25^h.$$

Since $(25/24)/(3/4)^2 = 50/27$, these inequalities imply (8). Exchanging A, B leaves that assertion unchanged. □

Write $(x_l, t_l) = (X(A_l), T(A_l))$ for $l \in L$, and $(y_p, u_p) = (X(A_p), T(A_p))$ for $p \in P$. Every t_l, u_p is positive, and each family has distinct second coordinates. Equation (7) gives, for each edge,

$$y_{p(e)} - x_{l(e)} > 0, \qquad u_{p(e)} - t_{l(e)} = (y_{p(e)} - x_{l(e)})^2. \tag{9}$$

For two different labels in either one class, put $\Delta x = X(B) - X(A)$ and $\Delta t = T(B) - T(A)$. Lemma 3.1 implies

$$-\frac{(\Delta x)^2}{2} \pm \frac{\Delta t}{4} < -\frac{(\Delta x)^2}{27} < 0. \tag{10}$$

The constant $50/27 < 2$ is what supplies a strict support margin.

4. Planar seeds with strict support margins

For a positive parameter ε, define

$$C(x,t) = x^2 + t/2, \qquad\qquad D(y,u) = -y^2 + u/2,$$

$$a_\varepsilon(x,t) = \varepsilon x + i\big(-1/2 + \varepsilon^2 C(x,t)\big), \tag{11}$$

$$b_\varepsilon(y,u) = -\varepsilon y + i\big(1/2 + \varepsilon^2 D(y,u)\big). \tag{12}$$

The L seeds are $a_\varepsilon(x_l, t_l)$ and the P seeds are $b_\varepsilon(y_p, u_p)$. For fixed d all coefficients in these formulas are fixed, and every asymptotic estimate in ε below is taken as $\varepsilon \to 0^+$. Its constant may depend on the finite incidence pattern.

Lemma 4.1 (Seed identities). *For arbitrary real x, t, y, u, direct expansion gives*

$$\sigma(a_\varepsilon(x,t), a_\varepsilon(y,u)) = \varepsilon^2\left(-\frac{(y-x)^2}{2} - \frac{u-t}{4}\right) + \varepsilon^4 C(x,t)\big(C(y,u) - C(x,t)\big), \tag{13}$$

$$\sigma(b_\varepsilon(x,t), b_\varepsilon(y,u)) = \varepsilon^2\left(-\frac{(y-x)^2}{2} + \frac{u-t}{4}\right) + \varepsilon^4 D(x,t)\big(D(y,u) - D(x,t)\big), \tag{14}$$

$$|a_\varepsilon(x,t)|^2 = \frac{1}{4} - \frac{\varepsilon^2 t}{2} + \varepsilon^4 C(x,t)^2, \tag{15}$$

$$|b_\varepsilon(y,u)|^2 = \frac{1}{4} + \frac{\varepsilon^2 u}{2} + \varepsilon^4 D(y,u)^2. \tag{16}$$

If $u - t = (y-x)^2$, then

$$|b_\varepsilon(y,u) - a_\varepsilon(x,t)|^2 = 1 + \varepsilon^4\big(D(y,u) - C(x,t)\big)^2. \tag{17}$$

Proof. Substitute (11)–(12) into (2) and collect powers of ε for the first four identities. For the last identity the horizontal difference is $-\varepsilon(x+y)$, and the vertical difference is $1+\varepsilon^2(D(y,u)-C(x,t))$. The coefficient of ε^2 in the squared distance is

$$(x+y)^2 + 2(D(y,u) - C(x,t)) = u - t - (y-x)^2 = 0.$$

□

Within either seed class, the leading support coefficient is strictly negative by (10). Across the two classes, the seeds tend to $-i/2$ and $i/2$, and both ordered support quantities tend to $-1/2$. Thus all ordered supports between different labels are negative for sufficiently small positive ε.

The squared-norm expansions also show that the seed radii are distinct. Within one class, the coefficient of ε^2 in their difference is a nonzero multiple of the difference of the second coordinates. Across the classes, the coefficient is $-(t_l + u_p)/2 \neq 0$ in $|a_\varepsilon(x_l, t_l)|^2 - |b_\varepsilon(y_p, u_p)|^2$. All seeds have norms tending to $1/2$, so their norms lie strictly between zero and one for small ε.

The next step makes the incident distances exactly one while spending only a third-order rotational error. This is smaller than the second-order support margin within a seed class.

5. Correcting the unit distances

Fix an incident pair, abbreviate $a = a_\varepsilon(x,t)$, $b = b_\varepsilon(y,u)$, and assume $x < y$ and $u - t = (y-x)^2$. Put

$$M = D(y,u) - C(x,t), \qquad q = \overline{a}b,$$
$$\tau = \frac{x-y}{2} + \varepsilon^2\big(xD(y,u) + yC(x,t)\big).$$

Then $\operatorname{Im} q = \varepsilon\tau$, $q \to -1/4$, and $\tau \to (x-y)/2 < 0$. Define

$$F = \tau^2 - \varepsilon^2 M^2 \operatorname{Re} q - \frac{\varepsilon^6 M^4}{4}, \tag{18}$$
$$z = \operatorname{Re} q + \frac{\varepsilon^4 M^2}{2} - i\varepsilon\sqrt{F}, \qquad w = \frac{z}{q}. \tag{19}$$

For sufficiently small positive ε, $F > 0$ and $q \neq 0$, so these expressions are well defined.

Lemma 5.1 (Exact small rotation)**.** *The rotation in* (19) *satisfies*

$$|w| = 1, \qquad |a - bw| = 1, \qquad \operatorname{Im}(\overline{a}bw) < 0, \qquad w - 1 = O(\varepsilon^3). \tag{20}$$

Proof. The definition of F gives

$$|z|^2 = \left(\operatorname{Re} q + \frac{\varepsilon^4 M^2}{2}\right)^2 + \varepsilon^2 F$$
$$= (\operatorname{Re} q)^2 + \varepsilon^2\tau^2 = |q|^2,$$

so $|w| = 1$. Since $qw = z$ and (17) holds,

$$|a - bw|^2 = |a|^2 + |b|^2 - 2\operatorname{Re}(qw)$$
$$= |a-b|^2 - \varepsilon^4 M^2 = 1.$$

Also $\operatorname{Im}(qw) = -\varepsilon\sqrt{F} < 0$.

To obtain the order of the correction, rationalize the imaginary difference. The exact identity is

$$w - 1 = \frac{\varepsilon^3}{q}\left(\frac{\varepsilon M^2}{2} + i\frac{M^2(\operatorname{Re} q + \varepsilon^4 M^2/4)}{\sqrt{F} - \tau}\right). \tag{21}$$

Indeed, its real part before division by q is $\varepsilon^4 M^2/2$, while its imaginary part follows from

$$(\sqrt{F} + \tau)(\sqrt{F} - \tau) = -\varepsilon^2 M^2(\operatorname{Re} q + \varepsilon^4 M^2/4).$$

The denominators in (21) tend respectively to $-1/4$ and $y - x > 0$. The coefficient of ε^3 is therefore continuous near zero and bounded, proving the estimate. $\square$

Proposition 5.2 (Prepared seeds)**.** *For the incidence pattern in Section 3, there are seeds $(b_i)_{i \in L \sqcup P}$ and rotations $(w_e)_{e \in E}$ satisfying conditions (1), (3), and (4) of Proposition 2.3, as well as*

$$|w_e| = 1, \qquad \operatorname{Im}(\overline{b_{l(e)}} b_{p(e)} w_e) < 0 \quad (e \in E).$$

Proof. Apply Lemma 5.1 to every incident pair. Since the edge set is finite,

$$H(\varepsilon) := \sum_{e \in E} |w_e(\varepsilon) - 1| = O(\varepsilon^3).$$

For each ordered pair in the same seed class, (13) or (14) has the form $-c\varepsilon^2 + O(\varepsilon^4)$ with $c > 0$. Consequently that support is less than $-2H(\varepsilon)$ for sufficiently small positive ε. Across the two classes, the support tends to $-1/2$, so the same inequality holds. The norm conditions follow from Section 4. There are finitely many pairs and edges, so one sufficiently small positive ε satisfies every required condition simultaneously. Fix such an ε. $\square$

Prepared seeds do not yet guarantee distinct subset products of the rotations. We obtain that last property by varying their radii.

6. Separating products by a radius perturbation

For positive r, s with

$$-1 < c(r,s) := \frac{r^2+s^2-1}{2rs} < 1,$$

define $\alpha(r,s) = \arccos c(r,s) \in (0,\pi)$. The law of cosines gives

$$|r - se^{-i\alpha(r,s)}| = 1. \tag{22}$$

The strict inequalities define an open region on which α is smooth.

Lemma 6.1 (Nonvanishing mixed derivative)**.** *On this region,*

$$\frac{\partial^2\alpha}{\partial s\,\partial r}(r,s) = \frac{1}{r^2s^2(1-c(r,s)^2)^{3/2}} > 0. \tag{23}$$

Proof. Differentiation gives

$$c_r = \frac{r^2-s^2+1}{2r^2s}, \qquad c_s = \frac{s^2-r^2+1}{2rs^2}, \qquad c_{rs} = -\frac{r^2+s^2+1}{2r^2s^2}.$$

Using $\alpha_r = -c_r/(1-c^2)^{1/2}$, we obtain

$$\alpha_{rs} = -\frac{c_{rs}(1-c^2)+c\,c_rc_s}{(1-c^2)^{3/2}}.$$

Substitution and cancellation give $-c_{rs}(1-c^2) - c\,c_rc_s = 1/(r^2s^2)$, proving the formula. $\square$

Lemma 6.2 (Generic separation)**.** *Let L, P, E, l, p be as in Proposition 2.3, and fix real phases θ_e for $e \in E$. Let $\mathcal{U}$ be a nonempty open subset of $\mathbb{R}^L \times \mathbb{R}^P$ on which all radii r_l, s_p are positive and all incident pairs satisfy $-1 < c(r_{l(e)}, s_{p(e)}) < 1$. Set*

$$w_e(r,s) = \exp\big(i(\theta_e - \alpha(r_{l(e)}, s_{p(e)}))\big). \tag{24}$$

There is a point of $\mathcal{U}$ at which all products $\prod_{e\in S} w_e(r,s)$, $S \subseteq E$, are distinct.

Proof. Fix different subsets $S, T \subseteq E$ and put $a_e = \mathbf{1}_S(e) - \mathbf{1}_T(e)$. At least one a_e is nonzero. Equality of their products is equivalent to

$$\Phi(r,s) := \sum_{e\in E} a_e\big(\theta_e - \alpha(r_{l(e)}, s_{p(e)})\big) \in 2\pi\mathbb{Z}. \tag{25}$$

The equality set is relatively closed in $\mathcal{U}$, because the products are continuous.

It has empty interior. Otherwise it contains a nonempty open ball within $\mathcal{U}$. On that connected ball, the continuous function Φ takes values in the discrete set $2\pi\mathbb{Z}$, hence is constant. Choose e_0 with $a_{e_0} \neq 0$. Differentiating with respect to $r_{l(e_0)}$ and $s_{p(e_0)}$ kills every summand except the one for e_0: injectivity of the endpoint pair prevents any other edge from depending on both these coordinates. Thus on the ball

$$\frac{\partial^2\Phi}{\partial s_{p(e_0)}\,\partial r_{l(e_0)}} = -a_{e_0}\alpha_{rs}(r_{l(e_0)}, s_{p(e_0)}) \neq 0$$

by Lemma 6.1, contradicting constancy.

The complement of each equality set is therefore relatively open and dense. There are finitely many pairs $S \neq T$, so their complements have nonempty intersection in $\mathcal{U}$. Explicitly, intersecting one such complement with any nonempty relatively open subset leaves a nonempty relatively open subset; repeat this for the finite list. Any point in the resulting intersection works. □

Proposition 6.3 (Separated seeds). *The prepared seeds of Proposition 5.2 can be perturbed to satisfy every hypothesis of Proposition 2.3.*

Proof. Keep the directions of all prepared seeds fixed. Write

$$\eta_i = \frac{b_i}{|b_i|}, \qquad \widetilde{b}_l = r_l \eta_l \ (l \in L), \qquad \widetilde{b}_p = s_p \eta_p \ (p \in P).$$

Choose θ_e with $e^{i\theta_e} = \eta_{l(e)}/\eta_{p(e)}$, and use the rotations in (24). For every feasible radius vector, (22) gives

$$|\widetilde{b}_{l(e)} - \widetilde{b}_{p(e)} w_e(r, s)| = |r_{l(e)} - s_{p(e)} e^{-i\alpha(r_{l(e)}, s_{p(e)})}| = 1. \tag{26}$$

We first check that at the original radius vector these formulas recover the prepared rotations. Put $r = |b_{l(e)}|$, $s = |b_{p(e)}|$, and $q_e = \overline{b_{l(e)}} b_{p(e)} w_e$ for a prepared rotation. Its norm is rs; the unit-distance identity gives $\operatorname{Re} q_e = (r^2 + s^2 - 1)/2$; and its imaginary part is negative by Proposition 5.2. In particular $|\operatorname{Re} q_e| < rs$, so the radius pair is strictly feasible. These properties determine

$$q_e = rs\, e^{-i\alpha(r,s)}.$$

Dividing by $\overline{b_{l(e)}} b_{p(e)}$ proves that the rotation agrees with (24).

All required inequalities at the original radius vector are strict: the radii lie between zero and one, are pairwise distinct, the incident chords are feasible, and each ordered support has the required margin. The seed and rotation formulas are continuous there. Finiteness of all label sets therefore gives an open neighborhood $\mathcal{U}$ on which these conditions persist. In particular, throughout this neighborhood,

$$\sigma(\widetilde{b}_i, \widetilde{b}_j) < -2 \sum_{e \in E} |w_e(r, s) - 1| \qquad (i \neq j).$$

Apply Lemma 6.2 inside $\mathcal{U}$ and choose a radius vector with distinct subset products. The norm of every rotation remains one, and (26) keeps each incident distance exactly one. The resulting seeds and rotations have all the desired properties. □

7. Completion of the construction

Proof of Theorem 1.1. Fix $d \geq 2$ and take the coordinate-incidence pattern from Section 3. Propositions 5.2 and 6.3 supply seeds and rotations to which Proposition 2.3 applies. By (6), the resulting convex set has

$$|V_d| = 2d^d 2^{d^{d+1}} = N_d$$

and at least

$$d^{d+1} 2^{d^{d+1}-1} = \frac{d}{4}\left(2d^d 2^{d^{d+1}}\right) = \frac{d}{4} N_d$$

unit pairs. The usual real-linear isometry $\mathbb{C} \to \mathbb{R}^2$ preserves cardinality, distances, and convex hulls, so this is the required Euclidean-plane configuration. □

The choices of a small parameter and a generic radius vector are existence arguments over nonempty open sets. The theorem specifies the number of vertices and a lower bound on unit pairs; it does not supply an exact rational coordinate list for V_d. Numerical approximations of the construction are not needed for the conclusion.

8. Lean correspondence and verification evidence

The submitted formalization [3] develops the proof in `Bounty.Proof`. Its theorem `plane_counterexample` states the construction in the literal Euclidean plane. The declaration `large_convex_counterexample` handles every proposed real constant and every cardinality threshold. Together they prove `erdos_96_false`: $U_c(n)$ is not $O(n)$.

The final declaration `Bounty.target` proves the negation of the exact supplied task proposition for `Erdos96.erdos_96`.

The source defines convex independence by excluding each point from the convex hull of the others. It counts unordered pairs at Euclidean distance one, and its extremal function is the supremum of the possible natural-number counts. The proof establishes a finite upper bound on those counts before using the supremum. The complex-plane construction is transported by a real-linear isometry to `EuclideanSpace` over $\mathbb{R}$ with two coordinates.

Argument in this paper	Submitted declaration
Strict radial support	`convexIndependent_of_support`
Convexity after amplification	`rotatedCopy_convexIndependent`
Counting amplified unit pairs	`amplification_unitPairs_bound`
Coordinate-incidence encoding	`coordinatePattern`
Digit separation	`digit_separation`
Exact small rotations	`seedCorrection_properties`
Prepared seed existence	`SeedPattern.exists_prepared`
Mixed derivative identity	`hasDerivAt_chordAngleLeft_right`
Generic subset-product separation	`exists_injective_subsetProducts`
Simultaneous radius perturbation	`PreparedSeed.exists_separated`
Theorem 1.1	`plane_counterexample`
Corollary 1.2	`large_convex_counterexample`
Negation of the linear bound	`erdos_96_false`
Exact submitted target	`Bounty.target`

The prose uses ordinary asymptotic notation for the small-parameter estimates. The Lean development expresses them through continuity and eventual inequalities near zero. It proves the nonconstancy used in Lemma 6.2 with a mixed finite difference on a coordinate rectangle and the one-variable mean value theorem, using the positive mixed derivative in (23). The argument above presents the equivalent local differentiation proof. The digit argument allows any injective numbering; the submitted code uses a finite-type enumeration.

8.1. Public source and recorded checks.

The public result record links the complete Lean proof and the machine-readable verification report. The source can also be retrieved as the `source` field of the solution API response. It contains 102,974 UTF-8 bytes with SHA-256

`d445f7ec23d57a36ea4402c11eb8cc65fd6fd18ca1d6192e17f41cc359ea308c`.

The public source was checked against the retained accepted submission on 14 September 2026 and is byte-identical. The public report identifies the full original report by SHA-256

`9381a6adba2b5a67c68e0fe5b53820eba063d6e10fd45e62c5d839220cde17dd`.

This digest commits to the original full report, rather than its public JSON projection.

Production verification and an isolated replay on 10 September 2026 accepted the proof. A further isolated replay on 14 September 2026 accepted the same proof and task in 188.879 seconds. Its report records successful statement comparison, dependency and permitted-axiom checks, and acceptance by Lean's default kernel. The verification permitted only `propext`, `Quot.sound`, and `Classical.choice`; the optional Nanoda checker was disabled. The challenge template contains an admitted target for comparison, but the submitted proof passes the dependency check without that admission.

The recorded environment uses Lean 4.33.1, with the following revisions:

Component	Revision
Mathlib	`0df444a360ea`
Formal Conjectures, patched	`8432eac99811`
Statement comparator	`68a064109f01`

The accompanying artifact directory retains the exact task bundle, source, replay report, and verifier-image identifier, with checksums and a command for repeating the isolated check when that image is available. The task commitment is

`1b64a17280cf8d5bb669a8a4bcef39b857c52b033a4f0a4d4355e409fbff4a6b`.

Independent finite arithmetic checks were also rerun on 14 September. They verified the digit margins and incidence counts for $d = 2, 3, 4$, the rational seed identities and supports for $d = 2, 3$, and the distinctness of all 1,024 counted edge labels for $d = 2$. These checks do not construct the generic perturbation or prove the infinite statement. That conclusion rests on the argument and its Lean proof. The kernel checks concern the formal target and its dependencies; they do not mechanically verify this prose exposition.

Acknowledgments

The submitted Lean source credits OpenAI Codex with substantial assistance in the mathematical exploration, construction, proofs, formalization, verification, and manuscript preparation. This exposition was prepared from that source with Codex assistance. The authors are responsible for the content.

References

[1] T. F. Bloom, *Erdős Problem 96*, Erdős Problems, `https://www.erdosproblems.com/96`. The formal statement and definitions used here are those in the pinned Formal Conjectures source identified in Section 8.

[2] A. Khopkar, *Edge complexity of geometric graphs on convex independent point sets*, arXiv:1605.08066v2, 21 April 2017 (first version 24 May 2016). `https://arxiv.org/abs/1605.08066v2`.

[3] L. Kruer and J. Kohlmeyer, *Superlinearly many unit distances in convex position*, Lean source submitted to Conjectures, 10 September 2026. Submission `7ffd1c09-2912-4c4c-a7ff-a4faa263f517`. Public source and attribution.