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GRACEFUL LABELINGS OF TREES WITH ONE VERTEX OF DEGREE FOUR AND ONE OF DEGREE THREE

Abstract. The accepted Lean submission proves that every finite tree having exactly one vertex of degree 4, exactly one vertex of degree 3, and all other vertices of degree at most 2 has a graceful labeling. This settles one of the three shapes of trees with exactly five leaves, in the direction of the Ringel–Kotzig Graceful Tree Conjecture. Each such tree is a *double spider*: a path joining the two branch vertices, with three paths hanging from one end and two from the other. The labelings are assembled by gluing an α-labeled path to a graceful spider whose gluing vertex carries label 0. One family, in which all five pendant paths have length two, receives an explicit closed formula. We state the reduction, the gluing lemmas, the label formulas and the case split, and identify the longer formal inputs by declaration and line. This paper is an exposition of the accepted artifact and a guide to its complete formal proof.

1. The question and the result

Let T be a finite tree with q edges, hence $q+1$ vertices. A *graceful labeling* of T is an injective map $f\colon V(T) \to \{0, 1, \dots, q\}$ such that the edge differences $|f(x) - f(y)|$, $xy \in E(T)$, are exactly $1, 2, \dots, q$, each once. The Ringel–Kotzig conjecture, also called the Graceful Tree Conjecture, asserts that every tree has a graceful labeling [1]. It remains open in general.

Theorem 1.1 (Main result)**.** *Let T be a finite tree with distinct vertices u and v such that* $\deg u = 4$, $\deg v = 3$ *and* $\deg w \le 2$ *for every other vertex w. Then T has a graceful labeling.*

No restriction is placed on the lengths of the paths in T. Such a tree has exactly five leaves. Indeed, for any finite tree, $\sum_x (\deg x - 2) = 2q - 2(q+1) = -2$. Here the vertices of degree two contribute nothing, u and v contribute $2 + 1 = 3$, and each leaf contributes -1, so there are five leaves. The source records this count at line 58.

The same identity shows that a tree with exactly five leaves has branch vertices whose excess degrees $\deg x - 2$ sum to three. So the tree has one vertex of degree five, or one of degree four and one of degree three, or three of degree three. Theorem 1.1 treats exactly the middle shape.

1.1. Known results and scope.

The dynamic survey of Gallian [1] (edition of 6 November 2026) lists, among the trees known to be graceful, trees with at most four end-vertices [2], trees of diameter at most five, and spiders with at most five legs [3]. A spider has a single branch vertex, so five-legged spiders are the first shape of five-leaf trees above. We found no entry in that survey for the second shape, treated here. The public review record for this submission cites [3] as the source of the question. This is not an exhaustive literature search, and we make no priority claim.

Date: 16 November 2026.

Key words and phrases. graceful labeling, Graceful Tree Conjecture, α-labeling, trees with five leaves, Lean.

Submitted by JenW1N. Exposition and source guide prepared with AI assistance from the accepted Lean submission. The longer formal inputs are supplied by the linked formal source. This prose is not itself kernel-verified.

1.2. **The formal statement.** The statement checked by the service is the following, quoted from the task source. Here `edgeCount` and `degree` count edges and neighbours in a simple graph on `Fin n`. Edges are oriented by vertex order only so that each unordered edge is counted once.

```
def IsGraceful {n : ℕ} (G : SimpleGraph (Fin n)) : Prop :=
  ∃ f : Fin n → ℕ, Function.Injective f ∧
    (∀ v, f v ≤ edgeCount G) ∧
    (∀ d, 1 ≤ d → d ≤ edgeCount G →
      ∃! e : Fin n × Fin n, e.1 < e.2 ∧ G.Adj e.1 e.2 ∧ Nat.dist (f e.1) (f e.2) = d)

def Branch43 {n : ℕ} (G : SimpleGraph (Fin n)) (u v : Fin n) : Prop :=
  u ≠ v ∧ degree G u = 4 ∧ degree G v = 3 ∧
  ∀ w, w ≠ u → w ≠ v → degree G w ≤ 2

def Target10 : Prop := ∀ n : ℕ, ∀ G : SimpleGraph (Fin n),
  G.IsTree → (∃ u v, Branch43 G u v) → IsGraceful G
```

In words, every finite simple tree on vertex set $\{0, \dots, n-1\}$ with the stated degree pattern has a labeling that is injective, bounded by the number of edges, and realises each difference $1, \dots, q$ on exactly one edge. This is the definition of a graceful labeling of Section 1, so the formal statement is exactly Theorem 1.1.

2. Reduction to double spiders

For positive integers a, b, c, d, e, f let $D = D(a, b, c; d; e, f)$ be the tree obtained from a path $u = s_0, s_1, \dots, s_d = v$ by attaching three further paths, of a, b and c edges, at u, and two paths, of e and f edges, at v. Thus $\deg u = 4$, $\deg v = 3$, and D has $q = a+b+c+d+e+f$ edges. We call $s_0 \cdots s_d$ the *spine* and the five attached paths the *arms*.

The source fixes a vertex order. Vertex 0 is u, vertices $1, \dots, d$ run along the spine, and the arms follow in the order a, b, c, e, f. Each nonzero vertex j has one smaller neighbour, its parent,

$$\pi(j) = \begin{cases} 0, & j \in \{d+1,\ d+a+1,\ d+a+b+1\}, \\ d, & j \in \{d+a+b+c+1,\ d+a+b+c+e+1\}, \\ j-1, & \text{otherwise.} \end{cases}$$

This is `doubleSpiderParent`. Every vertex is joined to 0 by parents, so D is a tree.

Proposition 2.1 (Structural reduction)**.** *Every tree satisfying the hypotheses of Theorem 1.1 is isomorphic to $D(a, b, c; d; e, f)$ for some positive $a, \dots, f$.*

Idea. Take the unique path from u to v; its length is $d \geq 1$. No inner vertex of this path has a neighbour off the path, since its degree is at most two. Deleting u leaves four components and deleting v leaves three. All have maximum degree at most two, so the components away from the spine are paths ending in a leaf. This yields three arms at u and two at v, all of positive length. The source formalises this with a component analysis of $G - u$ and $G - v$ and an explicit isomorphism (line 9402). □

A graceful labeling transports along a graph isomorphism (`isGraceful_of_iso`, line 9478). The order of the arms at either branch vertex is also irrelevant. The source proves the needed reorderings as isomorphisms of the double-spider graphs. So it suffices to label every $D(a, b, c; d; e, f)$.

3. Tools: α-labelings and gluing

A graceful labeling f of a tree with q edges is an *α-labeling with cut k*, where $0 \leq k < q$, if every edge xy satisfies $\min(f(x), f(y)) \leq k < \max(f(x), f(y))$ [1]. Two relabelings preserve this property

and complement every difference $\delta \mapsto q+1-\delta$:

$$\rho_{q,k}(x) = \begin{cases} k-x, & x \le k, \\ q+k+1-x, & x > k, \end{cases} \qquad (\text{cut } k;\ \text{sends } 0 \mapsto k),$$

$$\nu_{q,k}(x) = \begin{cases} q-k+x, & x \le k, \\ x-k-1, & x > k, \end{cases} \qquad (\text{cut } q-k-1;\ \text{sends } k+1 \mapsto 0).$$

Since the differences $1, \dots, q$ are permuted, the results are again graceful. These are `IsAlphaLabeling.reverse` and `IsAlphaLabeling.normalizePin`. We also write $\mathrm{sh}_{k,m}(x) = x$ for $x \le k$ and $\mathrm{sh}_{k,m}(x) = x+m$ for $x > k$, which opens a gap of m unused labels just above the cut.

Lemma 3.1 (Gluing at an α-cut)**.** *Let a tree T be the union of two subtrees T_1, T_2 that share exactly one vertex w and no edge, with q_1 and q_2 edges. Suppose T_1 has an α-labeling f with cut k and $f(w) \in \{0, k\}$, and T_2 has a graceful labeling g with $g(w) = 0$. Then T is graceful.*

Proof. If $f(w) = 0$, replace f by $\rho_{q_1,k} \circ f$; now $f(w) = k$. Define $h = \mathrm{sh}_{k,q_2} \circ f$ on T_1 and $h = k+g$ on T_2. Both rules give k at w. On T_1 the low labels lie in $[0, k]$ and the high labels in $[k+1+q_2,\ q_1+q_2]$. The labels on T_2 lie in $[k, k+q_2]$, so the three ranges meet only at w. Every edge of T_1 crosses the cut, so its difference grows by exactly q_2, giving $q_2+1, \dots, q_2+q_1$. The edges of T_2 keep their differences $1, \dots, q_2$. □

This is `isGraceful_amalgamation` (line 6408), the familiar kind of gluing for α-labelings. We make no claim about its origin.

Lemma 3.2 (Appending a path)**.** *Let f be an α-labeling with cut k of a tree T with q edges, let w be a vertex with $x = f(w)$, and let $m \ge 1$ satisfy $m \ge k+1-x$ if $x \le k$ and $m \ge x-k$ if $x > k$. Let p be a graceful labeling of the path $y_0 y_1 \cdots y_{m-1}$, with labels $0, \dots, m-1$, such that*

$$p(y_0) = x+m-k-1 \quad (x \le k), \qquad p(y_0) = x-k-1 \quad (x > k).$$

Attach the path to T by the edge $w y_0$. Then $\mathrm{sh}_{k,m} \circ f$ on T together with $y_i \mapsto k+1+p(y_i)$ is a graceful labeling of the enlarged tree.

Proof. The label ranges $[0, k]$, $[k+1, k+m]$ and $[k+m+1, q+m]$ are disjoint. Old edges cross the cut and get differences $m+1, \dots, m+q$. Path edges keep $1, \dots, m-1$. The choice of $p(y_0)$ makes the joining edge have difference exactly m. □

The source proves this for a vertex labelled 0 (`NatParentCertificate.appendAtZero`) and for the end of the long arm of a spider (`NatSpiderCertificate.exists_append`). The following input supplies p.

Lemma 3.3 (Formal input: prescribed first label)**.** *For $0 \le a < m$, the path on m vertices has a graceful labeling whose first vertex has label a.*

This is `exists_graceful_path_endpoint` (line 5225). The source proves it by recursion on m, doubling paths with the labels $2x$ and $2s-1-2x$ (`binaryLabel`, `allEndpoints`).

3.1. **Alternating words and prescribed-zero paths.** Consider a word $W = w_0 w_1 \cdots w_{A+B+1}$ that uses each symbol $\mathsf{L}_0, \dots, \mathsf{L}_A$, $\mathsf{H}_0, \dots, \mathsf{H}_B$ exactly once, alternates between L- and H-symbols, and has the consecutive index sums equal, as a multiset, to $\{0, 1, \dots, A+B\}$. The source calls such a word, starting with L_1, *terminal* (`Terminal`, line 2163). Read it as a path $x_0 \cdots x_{A+B+1}$ with x_i named by w_i.

Lemma 3.4 (Reading off a word)**.** *The maps $\lambda(\mathsf{L}_i) = A-i$, $\lambda(\mathsf{H}_j) = A+1+j$ and $\mu(\mathsf{L}_i) = B+1+i$, $\mu(\mathsf{H}_j) = B-j$ are α-labelings of the path with cuts A and B respectively.*

Proof. Each map is a bijection onto $\{0, \dots, A+B+1\}$, with the L- and H-labels on opposite sides of the cut. An edge joins some L_i and H_j and has difference $i+j+1$ under both maps. The index sums are $0, \dots, A+B$, so the differences are $1, \dots, A+B+1$. □

Proposition 3.5 (Formal input: symbolic construction)**.** *Let $d \geq 1$ and $1 \leq c \leq 2d-16$. There is a terminal word of length $d+c$ whose symbol in position $d-1$ is L_A if d is odd and H_B if d is even.*

This is `symbolic_proposition7` (line 2567). The words are explicit. They start from $\mathsf{L}_1\mathsf{H}_0\mathsf{L}_0$, $\mathsf{L}_1\mathsf{H}_1\mathsf{L}_0\mathsf{H}_0$, $\mathsf{L}_1\mathsf{H}_0\mathsf{L}_0\mathsf{H}_2\mathsf{L}_2\mathsf{H}_1$ and $\mathsf{L}_1\mathsf{H}_0\mathsf{L}_0\mathsf{H}_2\mathsf{L}_2\mathsf{H}_1\mathsf{L}_4\mathsf{H}_3\mathsf{L}_3$. They grow by four-symbol blocks such as $\mathsf{H}_{B+2}\mathsf{L}_{A+2}\mathsf{H}_{B+1}\mathsf{L}_{A+1}$ and by six-symbol cycle moves on a pair of words. A final exit step joins the pair, in one of eight cases fixed by the parity of d and by $(c-1) \bmod 4$. Lemma 3.4 with λ or μ puts label 0 on position $d-1$.

Proposition 3.6 (Prescribed-zero α-paths)**.** *Let $a, b \geq 1$ with $(a,b) \neq (2,2)$. The path $x_0 \cdots x_{a+b}$ has an α-labeling with x_a labelled* 0*.*

Source. The source names this *Rosa's prescribed-zero alpha-path theorem* and reproves it (`rosa_zero_alpha_path`, line 4607). Reversing the path swaps a and b. If $\max(a,b) \leq 13$, one of 90 explicit labelings, checked by the Lean kernel, is used. If $a \geq 14$ and $b \leq a$, apply Proposition 3.5 with $(d,c) = (a+1, b)$; the condition $b \leq 2(a+1)-16$ holds. Then read the word with λ if $a+1$ is odd and with μ if $a+1$ is even. □

The exception is genuine. No α-labeling of the five-vertex path gives its middle vertex label 0, which is why arms of lengths $(2,2)$ need separate treatment below.

4. The labelings

The final theorem `all_double_spiders_graceful` (line 9578) splits on the two arms at v and then on the arms at u:

(1) $(e,f) \neq (2,2)$: Section 4.1;
(2) $(e,f) = (2,2)$ and not $a=b=c=2$: Section 4.2, after reordering the arms at u so that $(a,b) \neq (2,2)$;
(3) $a=b=c=e=f=2$: the closed formula of Section 4.3.

4.1. **The general case.** Split D at v. The first piece is the path through v formed by the arms e and f, with v in position e. The second is the four-legged spider $S(a,b,c,d)$ with centre u, whose leg of length d is the spine and ends at v. Proposition 3.6 gives the path an α-labeling with v labelled 0. By Lemma 3.1, it remains to show the following.

Proposition 4.1 (Four legs, leaf at zero)**.** *For all positive a,b,c,d the spider $S(a,b,c,d)$ has a graceful labeling in which the end of the leg of length d has label* 0*.*

Proof. Let $N = a+b+c$. Proposition 4.2 below gives a graceful labeling g of the three-legged spider $S(a,b,c)$ with centre label 0. On the path $z_0 \cdots z_d$ use the α-labeling with cut $k = \lfloor d/2 \rfloor$

$$p(z_j) = \begin{cases} (d-j)/2, & j \text{ even}, \\ d-(d-j)/2, & j \text{ odd}, \end{cases}$$

with integer division (`cutEndpointPath`). Consecutive values differ by j, and $p(z_0) = k$, $p(z_d) \in \{0, d\}$. Identify z_0 with the centre and label the old vertices by $k+g$ and z_j $(j \geq 1)$ by $\mathrm{sh}_{k,N}(p(z_j))$. Old differences are unchanged and the path differences become $N+1, \dots, N+d$; the label ranges meet only at the centre. If $p(z_d) = d$, replace every label x by $N+d-x$, which keeps the labeling graceful. In both cases the leaf z_d has label 0 (`graftPathLabel`, `fourLeafZero`, line 8116). □

Proposition 4.2 (Three legs, centre at zero)**.** *For all positive a,b,c the spider $S(a,b,c)$ has a graceful labeling with centre label* 0 *(`threeZero`, line 7907).*

Construction. Sort so that $a \le b \le c$. If $(a,b) \ne (2,2)$ and not $a=b=c$ even, take the α-labeling of the path formed by legs a and b with the centre labelled 0 (Proposition 3.6), with cut k. The source proves $k+1 \le c$. Then append leg c at the centre by Lemma 3.2 with $x=0$, $m=c$. If $a=b=2<c$, use one of nine kernel-checked labelings for $c \le 10$. For $c>10$ append $c-5$ vertices to a checked α-labeling of $S(2,2,5)$ with cut 4. If $a=b=c$ is even, label $S(a/2,a/2,a/2)$ recursively and subdivide every edge. With $s=3a/2+1$, the two vertices replacing an old vertex labelled y receive $2y$ and $2s-1-2y$, chosen according to depth parity (`NatParentCertificate.subdivide`, line 5356). □

4.2. **Two pendant arms of length two at the degree-three vertex.** Now $(e,f)=(2,2)$ and $(a,b) \ne (2,2)$. Split D at u. The first piece is the path formed by arms a and b, with u in position a; it receives an α-labeling with u labelled 0. The second is the spider $S(2,2,d+c)$ with centre v. Its long leg runs along the spine through u and continues as arm c, so u lies at distance d from the centre. By Lemma 3.1 it suffices to prove the following.

Proposition 4.3 (Pinned spiders)**.** *For all $d,c \ge 1$ the spider $S(2,2,d+c)$ has a graceful labeling with label 0 on the long-leg vertex at distance d from the centre (`spider_pinned_zero`, line 6540).*

Write $L=d+c$. Number $S(2,2,L)$ as in the source: 0 is the centre, $0{-}1{-}2$ and $0{-}3{-}4$ are the short legs, and $0{-}5{-}6{-}\cdots{-}(L+4)$ is the long leg, so the pin is vertex $4+d$. Call a labeling f *canonical with pin d* if it is an α-labeling with cut $\kappa = 2+\lfloor L/2 \rfloor$, if

$$(f(0),\dots,f(5)) = (1,\ L+4,\ 0,\ L+3,\ 2,\ L+1),$$

and if $f(4+d) = \kappa + (d \bmod 2)$. The map $\rho_{L+4,\kappa}$ or $\nu_{L+4,\kappa}$, according as $f(4+d)$ is κ or $\kappa+1$, then moves the pin label to 0. The proof has four regions.

First region, $c \le 2d-16$. Take the word W of Proposition 3.5, of length L. Label the centre and short legs by $1, L+4, 0, L+3, 2$, and the long-leg vertex at distance i by $\tau(w_{i-1})$, where

$$\tau(\mathsf{L}_i) = L+2-i, \qquad \tau(\mathsf{H}_j) = 3+j.$$

This is canonical with pin d (`pathSpiderLabel`, `canonical_pinned_first_region`).

Second region, $d \ge 8$ and $c \ge \lfloor d/2 \rfloor + 5$. For $d = 8,9,10,11$, a kernel-checked canonical labeling of $S(2,2,d+1)$ with pin d is supplied. Larger d follow from the prefix extension below, applied repeatedly (`canonical_one_tail`). After normalising the pin to 0, the cut k satisfies $k+1 \le \lfloor d/2 \rfloor + 4$ and $d+5-k \le \lfloor d/2 \rfloor + 4$. Lemma 3.2 then appends $c-1$ vertices to the long leg (`second_region_pinned`).

Lemma 4.4 (Prefix extension)**.** *If f is canonical on $S(2,2,L)$ with pin d, then*

$$f' = (1,\ L+8,\ 0,\ L+7,\ 2,\ L+5,\ 4,\ L+6,\ 3,\ f(5)+2,\ f(6)+2,\dots,f(L+4)+2)$$

is canonical on $S(2,2,L+4)$ with pin $d+4$ (`spiderPrefix`, `CanonicalPinnedSpider.prefixExtend`).

Small pins, $d \le 7$. Seven kernel-checked α-labelings of $S(2,2,d+c_0)$ with the pin already labelled 0 are supplied, where $c_0=2$ for $d=1$ and $c_0=1$ otherwise. When $c \ge 6,5,6,6,7,7,8$ for $d=1,\dots,7$ respectively, Lemma 3.2 appends $c-c_0$ vertices (`small_region_pinned`).

Finite remainder. The remaining 62 pairs, all with $d \le 12$ and $c \le 10$, use explicit kernel-checked labelings (`finiteCase_pinned`).

The four regions cover all $d,c \ge 1$. For instance, if $d \ge 8$ and $2d-16 < c < \lfloor d/2 \rfloor + 5$, then $d \le 12$ and $c \le 10$; such pairs belong to the finite remainder.

4.3. **All five pendant arms of length two.** For $D(2,2,2;d;2,2)$, with $q = d+10$, the source gives an explicit labeling ℓ in its vertex order (`allTwoLabel`, line 272). Put $r = d \bmod 2$, $t = 2\lfloor d/4 \rfloor$, $M = d+11-r$, and $\sigma = \mathit{true}$ when $d \equiv 0,3 \pmod 4$. On the spine, $0 \le j \le d$,

$$\ell(j) = \begin{cases} j/2, & j \equiv 0 \pmod 4, \\ d+\varepsilon-\lfloor j/2 \rfloor, & j \equiv 1 \pmod 4, \\ \lfloor j/2 \rfloor + 6, & j \equiv 2 \pmod 4, \\ d+\varepsilon'-\lfloor j/2 \rfloor, & j \equiv 3 \pmod 4, \end{cases} \qquad (\varepsilon,\varepsilon') = \begin{cases} (4,10), & d \text{ even}, \\ (10,4), & d \text{ odd}. \end{cases}$$

On the arms, vertices $d+1,\dots,d+10$ receive

$$d+10-r,\ 1,\ d+8-r,\ 3,\ d+6-r,\ 5, \text{ then } \begin{cases} t+2,\ M-t-2,\ t+4,\ M-t-4, & \sigma, \\ M-t-2,\ t+2,\ M-t-4,\ t+4, & \text{not } \sigma. \end{cases}$$

The difference on the edge from j to its parent is $\delta(j)$ (`allTwoDiff`). On the spine, $1 \le j \le d$,

$$\delta(j) = \begin{cases} d+5-j, & j \not\equiv d \pmod 2, \\ d+11-j, & j \equiv d \pmod 2,\ j \equiv 0,1 \pmod 4, \\ |d-1-j|, & j \equiv d \pmod 2,\ j \equiv 2,3 \pmod 4. \end{cases}$$

For $j = d+1,\dots,d+10$ the values are $d+10-r$, $d+9-r$, $d+8-r$, $d+5-r$, $d+6-r$, $d+1-r$ followed by $2,7,4,3$ if $d \equiv 0,1 \pmod 4$ and $4,9,2,5$ otherwise. The formal proof checks that ℓ is injective with values in $[0,q]$ and that δ is injective with values in $[1,q]$, by residue-class case analysis (`all_two_arms_graceful`, line 5490). Since there are q edges, δ is a bijection onto $\{1,\dots,q\}$.

5. A worked example

Figure 1 shows $D(1,2,2;2;1,3)$, with 11 edges. It falls under the general case. The path through v (arm e, v, arm f) receives the α-labeling $(4,0,3,1,2)$ with cut 1, the certificate for $(a,b) = (1,3)$. Proposition 4.1 labels $S(1,2,2,2)$ with the end of the spine leg at 0. Lemma 3.1 first reverses the path labels to $(2,1,3,0,4)$. It then shifts the high labels by 7 and adds 1 to the spider labels.

6. Formal verification and provenance

The submission was accepted by the service on 7 October 2026 at 22:47:43 UTC. Production verification run 44 completed at 23:02:23 UTC with status `VERIFIED`. The human-authorized reward review approved it on 8 October 2026. The public proof was retrieved for this paper and checked byte for byte against the accepted artifact.

The environment used Lean `v4.35.0-rc2` and Mathlib revision `065356127b1d`. The task source was a patched revision `4b69a7dca3e7` of Formal Conjectures, based on upstream `7d1a8c991274`. The run passed the statement comparator, task commitments, trusted-file checks, permitted-axiom check and Lean kernel check under `landrun+seccomp`. The allowed axioms were `propext`, `Quot.sound`, and `Classical.choice`. Nanoda was disabled. No new Lean replay was performed for this paper. JenW1N is the public solver name attached to the submitted proof.

The accompanying `validate.py` reimplements the label formulas of Sections 3–4 and checks gracefulness for all 117,649 double spiders with every length in $\{1,\dots,7\}$. It also checks further families reaching every case and region, and the all-two formula for $d \le 600$. These finite checks guard against transcription errors. Lemma 3.3 is instantiated there by search, not by the source's recursion. The theorem itself rests on the accepted Lean proof.

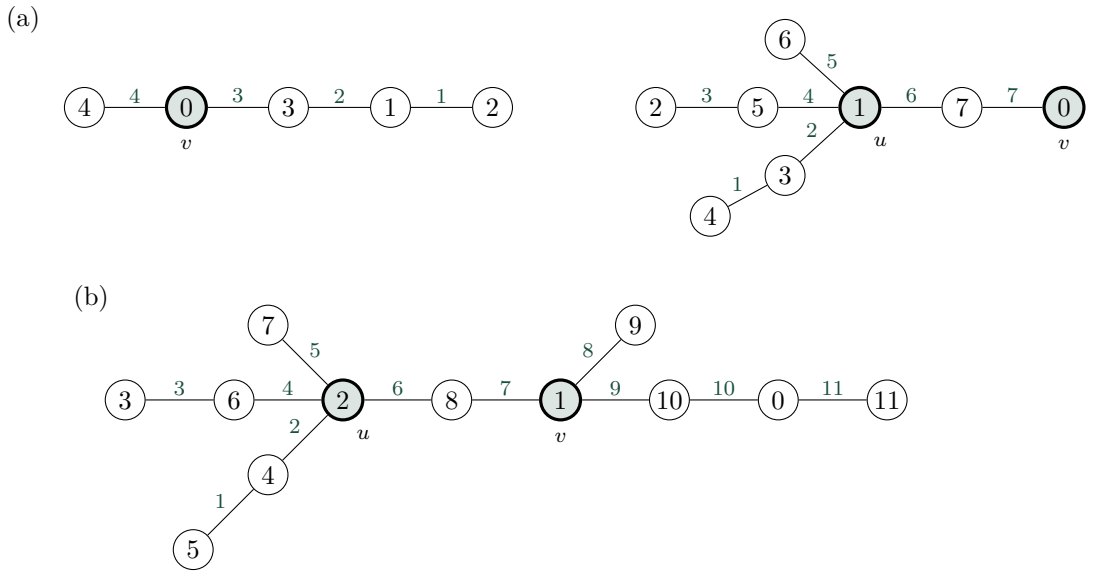


FIGURE 1. (a) The two pieces: the path through v with an α-labeling (cut 1, v labelled 0), and the spider $S(1,2,2,2)$ with the end of its spine leg labelled 0. (b) The glued graceful labeling of $D(1,2,2;2;1,3)$. Small numbers are edge differences; shaded vertices are u and v. All labels are computed and checked by `validate.py`.

7. Guide to the complete source

Line numbers below refer to the exact public submitted `Main.lean`, which has 9,608 lines. They count submitted bytes only, without the validator's trusted header and footer. Docstrings in the source refer to a "supplied manuscript" (Lemma 5, Proposition 7) that is not part of the public record; the names above follow the Lean declarations. The source checksum is

`c5ab8dc33b7b657b835e881db96c2c4a0635`
`c60747fb72049c3d3ce3eeca7761`.

Declaration and starting line	Mathematical role
`branch43_leaf_count` (58)	Five leaves.
`doubleSpiderParent` (245), `doubleSpider` (250)	The double spider $D(a,b,c;d;e,f)$ and its vertex order.
`allTwoLabel` (272), `allTwoDiff` (964)	Labels and differences for all arms of length two.
`all_two_arms_graceful` (5490)	Case 3.
`Terminal` (2163), `symbolic_proposition7` (2567)	Alternating words; Proposition 3.5.
`IsAlphaLabeling` (2697), `IsAlphaLabeling.reverse` (2729), `IsAlphaLabeling.normalizePin` (2789)	α-labelings, ρ and ν.
`spider` (2873), `spiderPrefix` (3011), `CanonicalPinnedSpider.prefixExtend` (3176)	$S(2,2,L)$; Lemma 4.4.
`appendSpiderLabels` (3567), `NatSpiderCertificate.exists_append` (5577)	Lemma 3.2 at the end of the long leg.

Declaration and starting line	Mathematical role
`boundaryLabel` (3711), `zeroAlphaPath_large_left` (3788)	Lemma 3.4; long prescribed-zero paths.
`pathSpiderLabel` (3414), `canonical_pinned_first_region` (3538)	First pinned region.
`zeroAlphaPath_small` (4496), `rosa_zero_alpha_path` (4607)	90 certificates; Proposition 3.6.
`binaryLabel` (4713), `allEndpoints` (5208), `exists_graceful_path_endpoint` (5225)	Lemma 3.3.
`NatParentCertificate.subdivide` (5356)	Subdivision step of Proposition 4.2.
`canonical_one_tail` (6129), `normalized_one_tail` (6145), `second_region_pinned` (6478)	Second pinned region.
`small_region_pinned` (6497), `finiteCase_pinned` (6198)	Small pins and finite remainder.
`spider_pinned_zero` (6540)	Proposition 4.3.
`isGraceful_amalgamation_at_cut` (6338), `isGraceful_amalgamation` (6408)	Lemma 3.1.
`doubleSpider_graceful_twoArms` (6969)	Case 2.
`doubleSpider_graceful_of_fourLeafZero` (7444)	Case 1.
`NatParentCertificate.appendAtZero` (7560), `ThreeZero.of_alpha_path` (7673)	Lemma 3.2 at a zero label; three legs from a path.
`threeZero22` (7734), `threeZero_symmetric` (7880), `threeZero` (7907)	Proposition 4.2.
`cutEndpointPath` (7927), `graftPathLabel` (7985), `NatParentCertificate.complement` (8089), `fourLeafZero` (8116)	Proposition 4.1.
`branch43_exists_doubleSpider_iso` (9402), `isGraceful_of_iso` (9478)	Proposition 2.1; transport.
`target10_of_doubleSpider_graceful` (9561)	Reduction of the formal statement to double spiders.
`all_double_spiders_graceful` (9578), `target` (9602)	Case split and exact committed target.

References

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