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# A RESTRICTED FOUR-SPEED LONELY-RUNNER SPECTRUM CLASSIFICATION, EXACT MAXIMA, AND A MATHEMATICAL ACCOUNT OF A LEAN PROOF

Formal proof credited to JenW1N

Abstract. For a finite set $V$ of positive integer speeds, let $\mathrm{ML}(V) = \max_t \min_{v\in V} \|vt\|$, where $\|x\|$ denotes distance to the nearest integer. We explain the classification of primitive four-element speed sets with pairwise greatest common divisors at most two and $\mathrm{ML}(V) < 1/4$. Every such set belongs to one of two explicit families or is $\{1, 3, 4, 14\}$. We derive the exact maxima of these families by interval estimates and a two-dimensional torus calculation. The resulting values have the form $s/(4s + k)$, with $1 \le k \le 4$ and $k \le s$; in fact the classification used here supplies $k = 1$ or $k = 2$. The argument is specific to the stated gcd restriction. This manuscript is an exposition of the cited formal submission, not a new claim that the unrestricted four-speed spectrum has been resolved.

## 1. Statement and normalization

Write $\|x\| = \min_{n\in\mathbb{Z}} |x - n|$ and

$$f_V(t) = \min_{v\in V} \|vt\|, \qquad \mathrm{ML}(V) = \max_{t\in\mathbb{R}/\mathbb{Z}} f_V(t).$$

The maximum exists by continuity and compactness. A set is *primitive* if the gcd of all its elements is one. Its elements are distinct because $V$ is a set.

**Theorem 1** (Restricted spectrum)**.** *Let $V \subset \mathbb{N}_{>0}$ have four elements. Suppose that*

$$\gcd(V) = 1, \qquad \gcd(u, v) \le 2 \quad (u, v \in V,\ u \ne v).$$

*Then either $\mathrm{ML}(V) \ge 1/4$, or there are integers $s, k$ with*

$$1 \le k \le 4, \qquad k \le s, \qquad \mathrm{ML}(V) = \frac{s}{4s + k}.$$

The central structural statement is stronger than the required spectral conclusion.

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*Date*: 18 October 2026.

Expository manuscript prepared from the submitted solution. Mathematical exposition reconstructed from the formal submission. A separate agent reviewed the mathematical exposition and found no substantive gap; its two clarity corrections are incorporated. This is not external peer review or a kernel check of the prose. The source is a Lean solution fragment in its original problem environment; this manuscript does not claim a fresh local verification of that environment.

**Theorem 2** (Classification below a quarter)**.** *Under the hypotheses of Theorem 1, if* $\mathrm{ML}(V) < 1/4$*, then one of the following holds:*

$$V = \{1, 2, 3, 4h\}, \qquad h \geq 1; \tag{I}$$

$$V = \{c - b, b, c, c + b\}, \qquad 0 < b < c, \quad \gcd(b, c) = 1; \tag{II}$$

$$V = \{1, 3, 4, 14\}. \tag{III}$$

This is a necessary classification under the hypotheses, not an assertion that every parameter in these displayed families gives an admissible four-element set. For example, (II) at $(c, b) = (2, 1)$ has only three distinct elements, and (I) with $3 \mid h$ violates the pairwise gcd assumption. The exact-maxima results below nevertheless hold for those parameter values as finite-set identities.

The original source defines the score on the additive circle and defines ML as a supremum. For these nonempty finite sets that definition agrees with the maximum above. The final formal declarations are `Bounty.Pure05.pure_classification`, `Bounty.Pure05.pure_target`, and `Bounty.target`. The theorem claimed here has exactly the positivity, cardinality, primitive, and pairwise gcd hypotheses appearing in that classification.

## 2. Exact maxima: the first family and the sporadic set

**Proposition 3.** *For every integer* $h \geq 1$*,*

$$\mathrm{ML}(\{1, 2, 3, 4h\}) = \frac{h}{4h + 1}.$$

*Proof.* Put $L = h/(4h + 1)$. Then $1/5 \leq L < 1/4$ and $4hL = h - L$. At $t = L$, the distances for speeds $1, 2, 3, 4h$ are respectively

$$L, \quad 2L, \quad 1 - 3L, \quad L,$$

all at least $L$. For the upper bound, symmetry $f_V(-t) = f_V(t)$ permits $0 \leq t \leq 1/2$. Suppose all four distances exceed $L$. The first three imply

$$t > L, \qquad 1 - 2t > L, \qquad |3t - 1| > L.$$

The branch $3t-1 > L$ contradicts $t < (1-L)/2$ because $L \geq 1/5$. Hence $t < (1-L)/3$. Consequently

$$-L < 4ht - h < \frac{4h(1 - L)}{3} - h = \frac{L}{3} \leq L,$$

where $4h(1 - L) = 3h + L$. Thus $\|4ht\| \leq L$, a contradiction. □

**Proposition 4.** $\mathrm{ML}(\{1, 3, 4, 14\}) = 4/17$.

*Proof.* At $t = 7/17$ the four distances are $7/17, 4/17, 6/17, 4/17$. For the converse restrict to $0 \leq t \leq 1/2$, and suppose all four distances exceed $4/17$. Then $t > 4/17$. The inequalities $|3t - 1| > 4/17$ and $|4t - 1| > 4/17$ force $t > 7/17$: otherwise the negative branch of the first gives $t < 13/51$, while the second gives either $t < 13/68$ or $t > 21/68$, both impossible. Also $|4t - 2| > 4/17$ gives $t < 15/34$. On this interval,

$$-4/17 < 14t - 6 < 3/17,$$

so $\|14t\| \leq 4/17$, a contradiction. □

## 3. The torus calculation for the additive family

Let $0 < b < c$ and $\gcd(b, c) = 1$. Write $V_{c,b} = \{c - b, b, c, c + b\}$ and

$$P(x, y) = \min\{\|x\|, \|y\|, \|x - y\|, \|x + y\|\}.$$

Then $f_{V_{c,b}}(t) = P(ct, bt)$. The function $P$ is unchanged by interchanging the coordinates, changing either coordinate's sign, or adding integers.

For $0 \le L \le 1/4$ define the closed triangle

$$A_L = \{(x, y) : y \ge L,\ x - y \ge L,\ x + y \le 1 - L\}.$$

Its vertices are

$$(2L, L), \quad (1 - 2L, L), \quad (1/2, 1/2 - L).$$

Every point of this triangle has $P(x, y) \ge L$. Conversely, choose representatives in $[-1/2, 1/2]$, take absolute values and order them as $0 \le y \le x \le 1/2$. The conditions $P \ge L$ give precisely the three displayed inequalities. Reflection $x \mapsto 1-x$ preserves $A_L$. Undoing the sign choices using this reflection shows that, modulo integer translates, the feasible locus is the union of $A_L$, its coordinate transpose, and their simultaneous negatives. In particular no feasible point can have $L > 1/4$.

**Lemma 5** (Orbit criterion)**.** *A torus point $(x, y)$ lies on the orbit $t \mapsto (ct, bt)$ if and only if $bx - cy \in \mathbb{Z}$.*

*Proof.* Necessity follows by cancellation. For sufficiency choose integers $p, q$ with $cp + bq = 1$. If $bx - cy = z \in \mathbb{Z}$, put $m = zq$ and $n = -zp$. Then $bm - cn = z$, so $b(x - m) = c(y - n)$. The real number $t = (x - m)/c$ gives $ct \equiv x$ and $bt \equiv y$ modulo one. □

The image of $A_L$ under $bx - cy$ is the interval

$$I_1(L) = \left[\frac{b - c}{2} + cL,\ b - (c + 2b)L\right],$$

and the image of its transpose is

$$I_2(L) = \left[-c + (2c + b)L,\ (b - 2c)L\right].$$

These endpoints follow by evaluating the linear functional at the three vertices; its image fills the entire interval by convexity. The negative triangles give the negatives of these intervals and therefore do not change whether an integer is present. The orbit criterion proves

$$\exists t\ f_{V_{c,b}}(t) \ge L \quad \Longleftrightarrow \quad I_1(L) \cap \mathbb{Z} \ne \varnothing \text{ or } I_2(L) \cap \mathbb{Z} \ne \varnothing. \tag{1}$$

Let $r_1$ and $r_2$ be the representatives in $\{0, 1, 2, 3\}$ of $2b - c$ and $b - 2c$ modulo four, and let $\bar{r} = (-r) \bmod 4$, also represented in that set. Define

$$G(c, b) = \min\left\{\frac{r_1}{c}, \frac{\bar{r}_1}{c + 2b}, \frac{r_2}{2c + b}, \frac{\bar{r}_2}{2c - b}\right\}. \tag{2}$$

**Proposition 6.**

$$\mathrm{ML}(V_{c,b}) = \frac{1 - G(c, b)}{4}.$$

*Proof.* Put $e = 1/4 - L \ge 0$. At $L = 1/4$ the two intervals collapse to their centers $(2b - c)/4$ and $(b - 2c)/4$. Thus

$$I_1(L) = \left[\frac{2b - c}{4} - ce,\ \frac{2b - c}{4} + (c + 2b)e\right],$$
$$I_2(L) = \left[\frac{b - 2c}{4} - (2c + b)e,\ \frac{b - 2c}{4} + (2c - b)e\right].$$

An interval $[n + r/4 - Ae, n + r/4 + Be]$, with $A, B > 0$, contains an integer exactly when

$$\min\{r/A, \bar{r}/B\} \le 4e.$$

If $r = 0$ its center is already an integer. Otherwise the first possible integers are $n$ and $n + 1$, at distances $r/4$ and $(4 - r)/4$. Apply this observation to both intervals in (1): feasibility is exactly $G(c, b) \le 1 - 4L$. All denominators in (2) are positive. Also $0 \le G \le 1$: if $c \ge 3$, use $r_1/c \le 1$; the remaining case $c = 2, b = 1$ gives $G = 0$. Hence $L = (1 - G)/4$ is in $[0, 1/4]$, is attained, and is maximal. □

The following table evaluates (2). Even–even residue pairs are excluded by coprimality.

| $c \bmod 4$ | $b \bmod 4$ | $G(c,b)$ |
|---|---|---|
| 0 | 1 | $1/(2c+b)$ |
| 0 | 3 | $\min\{2/(c+2b), 1/(2c-b)\}$ |
| 1 | 0 | $1/(c+2b)$ |
| 1 | 1 | $1/(2c-b)$ |
| 1 | 2 | 0 |
| 1 | 3 | $1/(2c+b)$ |
| 2 | 1 or 3 | 0 |
| 3 | 0 | $2/(2c+b)$ |
| 3 | 1 | $\min\{1/(c+2b), 1/(2c-b)\}$ |
| 3 | 2 | 0 |
| 3 | 3 | $1/(2c+b)$ |

For example at $(c,b) \equiv (0,3)$, $r_1 = 2, r_2 = 3$, so the four terms are $2/c, 2/(c+2b), 3/(2c+b), 1/(2c-b)$. The first is dominated by the second, and the third by the fourth because $b < c$. All other rows follow by the same multiplication of positive denominators. Each nonzero entry selected by the minimum is either $1/D$ with $D \equiv 1 \pmod 4$ and $D \geq 5$, or $2/D$ with $D \equiv 2 \pmod 4$ and $D \geq 10$. Write $D = 4s + k$, with $k = 1$ or 2. Then $s \geq k$ and

$$\frac{1 - k/D}{4} = \frac{s}{4s+k}.$$

A zero gap gives exactly $1/4$.

## 4. Quarter-good times and the parity reduction

Write $\mathcal{Q}_v(t)$ for $\|vt\| \geq 1/4$, equivalently

$$vt \in \bigcup_{j \in \mathbb{Z}} [j + 1/4, j + 3/4].$$

Call a tuple *quarter-free* if there is no common quarter-good time. Every set with ML $< 1/4$ is quarter-free. For an even speed, translating $t$ by $1/2$ preserves $\mathcal{Q}_v$. For an odd speed,

$$\|v(t + 1/2)\| = 1/2 - \|vt\|\,;$$

therefore at least one of $t, t + 1/2$ is quarter-good.

**Lemma 7.** *Three pairwise coprime positive integer speeds have a common quarter-good time.*

*Proof.* If all are odd use $t = 1/2$. Otherwise exactly one, say $e$, is even, and the other two are odd, say $u, v$. If $e \equiv 2 \pmod 4$, use $1/4$. Suppose $4 \mid e$. If one odd speed, say $v$, is at least three, choose $z$ with

$$ez \equiv (v-1)/2 \pmod v.$$

At $t = 1/4 + z/v$, both $v$ and $e$ are quarter-good: the former is at distance $1/4$, and the latter has residue $(v-1)/(2v) \in [1/4, 3/4]$. These two remain quarter-good at $t + 1/2$; choose the one of the two times good for $u$. If both odd speeds are one, use $t = 1/2 + 1/(4e)$. □

A quarter-free four-element primitive set satisfying the gcd bound must contain a multiple of four, since otherwise $1/4$ is good for all speeds. This multiple is unique, since two such speeds have gcd at least four. Primitivity guarantees an odd speed. There must be at least two odd speeds: with exactly one odd speed, halve the three even speeds. Their pairwise gcds become one, so Lemma 7 gives a common time for them; rescale time by $1/2$ and then choose a half-period translate good for the remaining odd speed. We have reduced the classification to:

(1) two odd speeds $a < b$ and two even speeds $E < F$, exactly one divisible by four;

(2) three odd speeds $a < b < q$ and one speed $E$ divisible by four.

Every gcd involving an odd speed is one, by the bound of two.

## 5. Rigidity of a finite cyclic interval

We isolate the finite-group observation used in the two-odd case.

**Lemma 8.** *Let $S$ be a cyclic interval of $L$ consecutive residues in $\mathbb{Z}/n\mathbb{Z}$, where $2 \le L < n-1$. If multiplication by a unit $u$ preserves $S$, then $u = 1$ or $u = -1$.*

*Proof.* For a step $d$, let $B_d = \{x \in S : x + d \notin S\}$. Then $|B_1| = 1$. Multiplication by $u$ gives a bijection $B_1 \to B_u$. Translating the interval does not affect these cardinalities, so take $S = \{0, \dots, L-1\}$. If $2 \le d \le n-2$, put $a = 0$ when $L \le d$ and $a = L - d$ otherwise. Both $a, a+1$ lie in $S$, while $a + d, a + 1 + d$ lie between $L$ and $n - 1$. Thus $|B_d| \ge 2$. A unit is nonzero, so only the steps $1, -1$ remain. □

For odd $n \ge 3$, the central half

$$C_n = \{x \in \mathbb{Z}/n\mathbb{Z} : n \le 4\bar{x} \le 3n\}$$

is a cyclic interval with the required size, except $n = 3$, for which the conclusion that its stabilizer consists of $\pm 1$ is immediate. Here $\bar{x}$ is the representative from 0 to $n - 1$.

**Lemma 9** (Grid rigidity)**.** *Let $n > 0$ be odd, let $v$ be odd, and assume $\gcd(n, u) = \gcd(n, v) = 1$. If no integer $j$ satisfies both*

$$\mathcal{Q}_u((2j+1)/n), \qquad \mathcal{Q}_v((2j+1)/(2n)),$$

*then $2u \equiv v$ or $2u \equiv -v \pmod{n}$.*

*Proof.* The case $n = 1$ is immediate. Put $w = (n - v)/2$, a unit modulo $n$. For odd $m$, the two positions $wm/n$ and $vm/(2n)$ sum to $m/2$, so their distances sum to $1/2$. Neither distance is $1/4$, because its denominator divides $2n$ and $n$ is odd. Thus the second displayed quarter condition is the complement of $wm \in C_n$. The assumed disjointness gives

$$u^{-1}C_n \subseteq w^{-1}C_n.$$

Every residue is represented by an odd integer, and multiplication by either unit is a bijection. The sets have equal cardinality, so they are equal. The unit $uw^{-1}$ stabilizes $C_n$; Lemma 8 gives $u = \pm w$, which is equivalent to the conclusion. □

## 6. Two odd speeds: congruences, intervals, and the final cases

Assume $a < b$ are odd, $E < F$ are even, one of $E, F$ is divisible by four, and the tuple is quarter-free with the pairwise gcd bound. Fix $n \in \{a, b\}$ and let $m$ be the other odd speed. On the grid $t = (2j+1)/(4n)$, speed $n$ is at distance $1/4$. If both even speeds were good, they would remain good after a half-period shift, and one of the two times would also be good for $m$. Hence the even speeds' good sets are disjoint on this grid. Applying Lemma 9 with the speed divisible by four divided by four and the other even speed divided by two gives

$$E/2 \equiv \pm F/2 \pmod{n}.$$

On the grid $(2j+1)/(2n)$, speed $n$ is at distance $1/2$, and the two even speeds have equal distances by this congruence. Another application of grid rigidity, now with $u = E/2$ and $v = m$, gives $E \equiv \pm m \pmod{n}$, and therefore also $F \equiv \pm m \pmod{n}$. Since $a, b$ are coprime and odd, the Chinese remainder theorem and the parity of $G = E, F$ yield

$$G = \epsilon a + \delta b + 2kab, \qquad \epsilon, \delta \in \{1, -1\}, \quad k \in \mathbb{Z}. \tag{3}$$

Indeed $ab$ divides $G - \epsilon a - \delta b$, and this difference is even while $ab$ is odd.

An interval of length at least $1/(2F)$ contains a quarter-good time for speed $F$, since every connected component of the complement of its quarter-good set has length $1/(2F)$ and open endpoints. If an interval inside $[0, 1/(4b)]$ is good for $E$ and has that length, select an $F$-good time in it and add $1/2$. Both odd speeds then become good, a contradiction. This proves

$$E \le 3b - 3. \tag{4}$$

In detail, if $E \ge 3b$, use $[1/(4E), 3/(4E)]$, of length $1/(2E) > 1/(2F)$. The only remaining even value above $3b-3$ is $E = 3b-1$. Use $[1/(4E), 1/(4b)]$. It is good for $E$, and its length $(E-b)/(4bE)$ exceeds $1/(2(E+2)) \ge 1/(2F)$ for odd $b \ge 3$, as follows by substitution $E = 3b - 1$.

For any positive lift (3), either $G = b - a$, $G = a + b$, or

$$G \ge 2ab - a - b. \tag{5}$$

To see this, negative $k$ makes $G$ negative, $k = 0$ gives the two positive base values, and $k \ge 1$ gives the inequality. When $a = 1$, the positive nonbase lift larger than $b + 1$ satisfies the sharper $G \ge 3b-1$, directly from the four residues $\pm 1 \pm b$ modulo $2b$. Combining these facts with (4) proves

$$E = b - a \quad \text{or} \quad E = a + b.$$

For $a \ge 3, b \ge 5$, a nonbase lift would give $2ab \le E + a + b \le a + 4b - 3$, whereas $2ab > a + 4b$. For $a = 1$, the four residue classes and $E \le 3b - 3$ leave only $b - 1, b + 1$.

6.1. **The interval supplied by an additive triple.** For coprime $0 < u \le v$ let $R = (v-u) \bmod 4$ and define

$$\eta(u, v) = \begin{cases} 0, & R = 0, \\ \min\{R/u, (4 - R)/v\}, & R > 0. \end{cases}$$

There is an interval of times, good for $u, v, u + v$, of length

$$\frac{1 - \eta(u, v)}{4(u + v)}. \tag{6}$$

Here is a direct construction. In the triangle $x, y \ge 1/4$, $x + y \le 3/4$, the orbit $(ut, vt)$ lies on the integer sections $vx - uy \in \mathbb{Z}$. Starting from such a point $(x, y)$ and moving in direction $(u, v)$ remains in the triangle for time $(3/4 - x - y)/(u + v)$. If $v - u = 4k + R$, the two nearest sections to $(1/4, 1/4)$ pass through

$$(1/4, 1/4 + R/(4u)), \qquad (1/4 + (4 - R)/(4v), 1/4).$$

Choose the one with the smaller increase of $x + y$, using the first with no increase when $R = 0$. The smaller normalized increase is at most one: this is immediate for $v \ge 3$ from $4 - R \le 3 \le v$, and for $v \le 2$ by inspection. Thus the chosen point lies in the triangle. Bezout's identity, as in Lemma 5, supplies its starting time. This proves (6). If $u, v, u + v, F$ are quarter-free, the interval length must be strictly less than $1/(2F)$, giving

$$F(1 - \eta(u, v)) < 2(u + v). \tag{7}$$

Applying this to the triple $a, b, E$ in its additive order gives the following bounds:

| Condition | Necessary inequality |
|---|---|
| $a \ge 3,\ E = b - a$ | $F < 4b$ |
| $E = a + b,\ a \equiv b \pmod 4$ | $F < 2(a + b)$ |
| $E = a + b,\ a \not\equiv b \pmod 4$ | $F(b - 2) < 2b(a + b)$ |
| $a = 1,\ b \ge 5,\ E = b - 1,\ b \equiv 1 \pmod 4$ | $F(b - 2) < 2b(b - 1)$ |
| $a = 1,\ b \ge 5,\ E = b - 1,\ b \equiv 3 \pmod 4$ | $F(b - 4) < 2b(b - 1)$ |

For the first row reorder the additive triple as $a, b-a, b$. Its two summands have opposite parity, each at least two; the smaller term in $\eta$ is at most $1/2$. For the second and third rows $\eta(a,b)$ is respectively 0 and $2/b$. For the last two rows use $u=1, v=b-1$, whose residues $R$ are respectively 3 and 1, so $\eta$ is $1/(b-1)$ or $3/(b-1)$.

6.2. **Completing the two-odd case.** If $a \geq 3$ and $E = b-a$, the lower bound (5) on a nonbase $F$ contradicts $F < 4b$, since $2ab-a-b \geq 4b$ for $a \geq 3, b \geq 5$. Thus $F = a+b$, giving family (II). If $a \geq 3$ and $E = a+b$, then $F$ is a large lift. In the equal-residue case, $2ab-a-b \geq 2(a+b)$ contradicts the table. In the different-residue case with $b \geq 7$,

$$(2ab-a-b)(b-2) - 2b(a+b) \geq (3b-1)(b-6) > 0.$$

The left side increases with $a \geq 3$, with coefficient $2b^2-7b+2 > 0$, proving the inequality by setting $a = 3$. Only $a = 3, b = 5$ remains. The bound gives $8 < F < 27$, and (3) leaves only $F = 22$. The tuple $(3, 5, 8, 22)$ has a common quarter-good time $7/15$, so this case is impossible.

Now let $a = 1$ and $b \geq 5$. If $E = b-1$, either $F = b+1$, giving family (II), or $F \geq 3b-1$. For $b \equiv 1 \pmod 4$, substituting $3b-1$ into the fourth row is already impossible for $b \geq 5$. For $b \equiv 3 \pmod 4$, substituting into the fifth row is impossible for $b \geq 11$. The only smaller case is $b = 7$; the table yields $8 < F < 28$, and the lift congruences give $F = 20$ or 22. The respective tuples $(1, 7, 6, 20)$ and $(1, 7, 6, 22)$ have witnesses $3/8$ and $1/4$. If instead $E = b+1$, then $F \geq 3b-1$. The equal-residue bound excludes $b \equiv 1 \pmod 4$, and the different-residue bound excludes $b \geq 11$. At $b = 7$, again $F = 20$ or 22; the former has $\gcd(8, 20) = 4$, and the latter tuple $(1, 7, 8, 22)$ has witness $1/3$.

Finally $a = 1, b = 3$. If $E = 2$, the unique multiple of four is $F = 4h$, giving family (I). If $E = 4$, the table gives $F < 24$. The gcd hypotheses require $F \equiv 2 \pmod 4$ and $3 \nmid F$, leaving $10, 14, 22$. Both 10 and 22 have witness $3/7$ with the other speeds $1, 3, 4$. The remaining set is (III). Every cited witness is checked by multiplying the four speeds by the rational time and reducing modulo one.

## 7. An odd-index floor-word lemma

**Lemma 10.** *Let $q$ be odd and $0 < \alpha < \beta < q/2$, with $\gcd(\alpha, q) = \gcd(\beta, q) = 1$. If*

$$\left\lfloor \frac{\alpha j}{q} \right\rfloor = \left\lfloor \frac{\beta j}{q} \right\rfloor \quad \text{for every positive odd integer } j \text{ with } 2j < q, \tag{8}$$

*then either $(\alpha, \beta) = (1, 2)$ or $(q, \alpha, \beta) = (13, 3, 4)$.*

*Proof.* For $q < 11$, the possible odd moduli are $3, 5, 7, 9$. The coprimality and size conditions, followed by the test $j = 3$ when it applies, leave only $(1, 2)$. Now let $q \geq 11$ and let $m$ be the largest odd integer below $q/2$. Thus $m \geq 5$ and $q = 2m+1$ or $2m+3$.

First $\beta = \alpha + 1$. Equality at $j = m$ gives $(\beta-\alpha)m < q$, so $\beta - \alpha \leq 2$. If the difference were two, put $\alpha m = Kq + r$, $0 \leq r < q$. For $q = 2m+1$, equality of the floors forces $r = 0$, contradicting coprimality and $m < q$. For $q = 2m+3$, it forces $1 \leq r \leq 2$. The identity

$$q\alpha = 2Kq + 3\alpha + 2r$$

then implies $\alpha = 2K+1$: the positive quantity $3\alpha + 2r$ is below $2q$. Thus $q = 3\alpha + 2r$, so $3\alpha < q < 3(\alpha+2)$, contradicting equality at $j = 3$.

Use the following elementary Farey construction. Start with $0/1 < \alpha/q < 1/1$. Whenever the two denominators sum to at most $m$, insert their mediant and retain the subinterval containing $\alpha/q$. The cross determinant stays one, each denominator stays at most $m$, and their sum increases, so the process terminates with

$$a/b < \alpha/q < c/e, \quad bc - ae = 1, \quad b, e \leq m, \quad b + e > m.$$

The reduced denominator $q > m$ prevents equality with a mediant or endpoint. For any determinant-one pair $a/b < d/f$, an interior fraction $p/q$ has positive integer coordinates $h = dq - fp$, $k = bp - aq$ and $q = bh + fk$; thus its denominator is at least the endpoint denominator sum. This verifies that the constructed pair has no fraction of denominator at most $m$ between its endpoints. Its successor on the right of $c/e$ is obtained explicitly by

$$K = \lfloor (m+b)/e \rfloor, \qquad d = Kc - a, \quad f = Ke - b.$$

These satisfy $0 < f \le m$, $ed - cf = 1$, and $e + f > m$. The same coordinate argument shows that they too are consecutive at order $m$.

For the interval just constructed, if $\beta/q < c/e$ as well, use positive integer coordinates

$$h = cq - e\alpha, \quad k = b\alpha - aq, \qquad q = bh + ek.$$

Since $\beta = \alpha + 1$ is still below $c/e$, $h \ge e + 1$. Therefore

$$q \ge be + b + e.$$

If $b, e \ge 2$, then $be \ge 2(b+e) - 4$, giving $q \ge 3m - 1 > 2m + 3$, a contradiction. An endpoint denominator one must be the left endpoint $0/1$. Indeed, if the right endpoint were $1/1$, the determinant equation would give $b = a + 1$, while $a/b < 1/2$ would force $a = 0, b = 1$, contrary to $b + 1 > m$. Its right neighbor is $1/m$. Thus $\beta m < q < 3m$, giving $\beta = 2, \alpha = 1$.

Otherwise $\alpha/q < c/e < \beta/q$. No reduced fraction of odd denominator at most $m$ can lie strictly between the two, by (8). Hence $e$ is even. Let $d/f$ be the next Farey fraction after $c/e$. Its denominator $f$ is odd, because $ed - cf = 1$, and similarly $b, c$ are odd from $bc - ae = 1$. Therefore $\beta/q < d/f$. We have

$$3 \le b, f \le m, \quad 2c < e, \quad b + e, f + e \ge m + 2, \quad e \mid b + f.$$

To exclude $b = 1$, its left endpoint would be zero, so $c = 1$ and the neighbor denominator condition would give $e = m$, impossible since $e$ is even. To exclude $f = 1$, its right endpoint would be one, so $e = c + 1$, contrary to $0 < 2c < e$. The strengthened sums use that $m, b, f$ are odd and $e$ even. The divisibility follows by adding the two determinant equations and using $\gcd(c, e) = 1$.

Put

$$h = cq - e\alpha, \qquad h' = e\beta - cq.$$

These are positive odd integers with $h + h' = e$. The determinant-one coordinates in the two adjacent intervals give positive integers $l, l'$ with

$$q = bh + el = fh' + el'.$$

Together with $q \le 2m + 3$ and the strengthened denominator sums, this gives

$$b(h-2) + 1 \le e, \qquad f(h'-2) + 1 \le e.$$

If either $h$ or $h'$ is one, the other inequality and $b, f \ge 3$ give $e \le 4$. Otherwise both are at least three; adding the bounds $3h \le e + 5$ and $3h' \le e + 5$ gives $e \le 10$. Thus $e = 4, 6, 8, 10$.

For $e = 4$, necessarily $c = 1$ and $b \equiv 1 \pmod 4$, so $b \ge 5$. The alternative $h = 3$ is impossible from $b(h-2) + 1 \le 4$. Hence $h = 1, h' = 3$, which forces $f = 3$, then $m = 5$ and $b = 5$. From $h + e\alpha = cq$ and $q = 11$ or $13$ we get $q = 13, \alpha = 3, \beta = 4$. For $e = 6$, both $h, h'$ equal three. The inequalities force $b, f \in \{3, 5\}$, but a determinant equation excludes three; thus $b = f = 5$, contrary to $6 \mid b + f$. For $e = 8$, the pair $(h, h')$ is $(3, 5)$ or $(5, 3)$; the coordinate equal to five forces its corresponding denominator below three, impossible. For $e = 10$, a coordinate seven is impossible, and the remaining pair $(5, 5)$ forces $b = f = 3$, contrary to $10 \mid b + f$. This proves the lemma. $\square$

## 8. Three odd speeds and the exceptional modulus

Let $a < b < q$ be odd and $4 \mid E$, with the tuple quarter-free and all gcds at most two. First $E < q$: if $E \geq q$, take $t = 1/(4E) + 1/2$. Speed $E$ has distance $1/4$; every odd speed $v \leq E$ has fractional part $1/2 + v/(4E)$, hence is also quarter-good. Since $E$ is a unit modulo $q$, there are unique unsigned representatives $r, s$ with

$$0 < r, s < q/2, \qquad a \equiv \pm Er, \quad b \equiv \pm Es \pmod{q}.$$

They are coprime to $q$. They are distinct: equality would imply $a \equiv \pm b \pmod{q}$; the plus case contradicts $0 < a < b < q$, and the minus case would give $a + b = q$, impossible since the left side is even and the right side odd.

**Lemma 11.** *After ordering $r < s$, the representatives satisfy the floor equalities (8).*

*Proof.* For $0 < y < q$ choose an integer $z$ with $Ez \equiv y \pmod{q}$, and consider $t = 1/4 + z/q$. Because $a$ is odd, direct reduction of $at$ shows that its quarter-good condition has the form

$$\mathcal{Q}_a(t) \quad \Longleftrightarrow \quad \big(P \iff 2(ry \bmod q) < q\big),$$

where the fixed truth value $P$ depends only on the sign in $a \equiv \pm Er$ and on $a \bmod 4$. Explicitly, with the plus sign $P$ is $a \equiv 1 \pmod 4$, and with the minus sign it is its negation. To check this, a phase $1/4$ is good exactly for residues in the lower half of the circle and a phase $3/4$ exactly for the upper half; negating a nonzero residue interchanges these halves. An analogous fixed value $Q$ works for $b, s$.

If $q \leq 4y \leq 3q$, speeds $E$ and $q$ are good at $t$ and remain so at $t + 1/2$. The first follows from $4 \mid E$ and $Et \equiv y/q$, and the second from $qt \equiv q/4$. The odd speeds $a, b$ cannot both be good at $t$. They also cannot both fail, because then both become good at $t + 1/2$. Thus their good conditions have opposite truth values.

For odd $j$ with $2j < q$, put $y = (q - j)/2$. This is a central residue as above. If

$$rj = kq + R, \qquad ry = lq + S, \qquad 0 < R < q,\ 0 \leq S < q,$$

then $2S + R = (r - k - 2l)q$. The coefficient on the right is either one or two. It follows that

$$2S < q \quad \Longleftrightarrow \quad r + k \text{ is odd.}$$

Therefore the opposite-truth-value condition becomes a fixed relation between the parities of $r + \lfloor rj/q \rfloor$ and $s + \lfloor sj/q \rfloor$. At $j = 1$ both floors are zero. Comparing each $j$ with this initial relation shows that the two floor values always have the same parity. Finally, on increasing an odd index by two, each floor increases by zero or one, because $2r, 2s < q$. Starting with equal zero values at $j = 1$, equal parity forces the same increment at every step. The floors are therefore equal. □

Lemma 10 gives $(r, s) = (1, 2)$ or $(q, r, s) = (13, 3, 4)$. The exceptional case cannot be quarter-free. Choose $z$ with $Ez \equiv 4 \pmod{13}$ and put $t = 1/2 + z/13$. Speed 13 is at distance $1/2$ and speed $E$ at distance $4/13$. For an odd speed congruent to $\pm 3E$ or $\pm 4E$ modulo 13, its position has the form $1/2 \pm 12/13$ or $1/2 \pm 16/13$ modulo one, with distances respectively $11/26$ and $7/26$, both greater than $1/4$. This is a common witness.

Thus the unsigned representatives are one and two. The unique odd number in $(0, q)$ congruent to $\pm E$ is $q - E$. The unique odd number there congruent to $\pm 2E$ is $|q - 2E|$. Consequently

$$V = \{E, q, q - E, |q - 2E|\}.$$

Let $x = q - E$. If $x < E$, this is family (II) with $(c, b) = (E, x)$; if $E < x$, use $(c, b) = (x, E)$. Equality is excluded by oddness of $q$. The parameters are coprime because $\gcd(E, q) = 1$. This completes the three-odd case, and with the parity reduction and two-odd argument proves Theorem 2.

## 9. Deducing the spectrum and delimiting the claim

*Proof of Theorem 1.* If $\mathrm{ML}(V) \geq 1/4$ there is nothing to prove. Otherwise apply Theorem 2. Proposition 3 supplies $(s, k) = (h, 1)$ in family (I), Proposition 4 supplies $(4, 1)$ in (III), and Proposition 6 with its residue table supplies $k = 1$ or 2 and $s \geq k$ in (II). These satisfy the stated range $1 \leq k \leq 4$. □

The conclusion does not say that every value allowed by the displayed formula occurs, nor does it classify sets with maximum at least $1/4$. It concerns four moving speeds relative to a stationary reference, with the precise arithmetic restrictions in Theorem 1. The unrestricted problem is outside the claim.

## Appendix A. Correspondence with the formal source

The source is the solution credited to JenW1N at

`https://conjectures.io/results/31ab8532-c253-4e14-95c7-be44878a02fa/solution`.

Line references below refer to the supplied 4,838-line solution fragment, `lonely-runner.lean`. They are navigation aids, not a claim that this prose was automatically extracted or verified. Declaration names are the stable identifiers.

| Mathematical component | Lean declaration and lines |
|---|---|
| Definitions and supremum estimates | `score_le_ML`, `ML_le_of_scores_le`, `exists_half_representative`, 36–82. |
| Family (I) | `familyOne_lower`, `familyOne_upper`, `familyOne_ML`, 84–187. |
| Sporadic set | `sporadic_lower`, `sporadic_upper`, `sporadic_ML`, 189–251. |
| Gap residues and table | `familyTwoGap`, `familyTwoGap_01` through `familyTwoGap_zero_right`, `familyTwoGap_spectrum`, 324–511. |
| Torus geometry | `pairScore_triangles`, `TriangleA.image_lower`, `orbit_of_integer_kernel`, 529–803. |
| Exact additive maximum | `quarter_integer_interval`, `familyTwo_gap_intervals`, `familyTwo_ML`, 805–1027. |
| Quarter predicates and classification | `Quarter`, `NoQuarter`, `PairwiseGcdTwo`, `Classified`, 1190–1204. |
| Odd floor words | `odd_floor_consecutive`, `ordinary_gap_classification`, `even_mediator_arithmetic`, `odd_floor_classification`, 1374–1880. |
| Cyclic rigidity | `cyclic_interval_stabilizer`, `centralHalf_stabilizer`, `grid_rigidity`, 1885–2325. |
| Additive triple interval | `triangle_section_interval`, `triangle_good_interval`, `triangle_fourth_bound`, 2431–2608. |
| Even-speed congruences | `even_grid_congruence`, `half_grid_congruence`, `two_even_additive_lifts`, 2857–3024. |
| Two-odd bounds and cases | `smaller_even_bound`, `two_odd_bounds`, `two_odd_classification`, 3045–3522. |
| Unsigned representatives | `exists_unsigned_scaled`, `odd_rep_one`, `odd_rep_two`, `classified_of_signed_one_two`, 3574–3732. |
| Three-odd floor conversion | `central_half_floor`, `floor_word_equal_of_parity`, `signed_grid_flag`, `three_odd_floor_words`, 3748–4165. |

| Mathematical component | Lean declaration and lines |
|---|---|
| Modulus thirteen | `thirteen_signed_quarter`, `thirteen_noQuarter_false`, 4174–4274. |
| Parity reduction | `three_coprime_quarter`, `two_odd_speeds`, `parity_shape`, 4286–4650. |
| Final assembly | `three_odd_classification`, `pure_classification`, `pure_target`, `target`, 4713–4834. |