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POLYNOMIAL MINIMAL GRAPHS OF DEGREE AT MOST FOUR ARE AFFINE

Abstract. Let u be a real polynomial in n variables, of total degree at most four, whose graph is a minimal hypersurface of $\mathbb{R}^{n+1}$. The accepted Lean submission proves that u is affine, for every n. We explain how the formal proof works. The top homogeneous component F of a nonaffine solution satisfies the cubic equation $|\nabla F|^2 \Delta F = \nabla F^{\mathsf{T}} D^2 F \, \nabla F$. A flux argument, applied to rescalings of the bounded divergence-free field $\nabla u/\sqrt{1+|\nabla u|^2}$, shows that F takes both signs. A spectral argument at a positive maximum and a negative minimum on the unit sphere shows that a homogeneous quartic solution of the cubic equation has only one sign. Degrees two and three, already excluded in the literature, are handled by short coefficient arguments. This paper is an exposition of the accepted artifact and a guide to its complete formal proof.

1. The question and the result

For a real polynomial $u \in \mathbb{R}[x_1, \dots, x_n]$ write u_i and u_{ij} for its partial derivatives, and put

$$\mathcal{M}(u) = \big(1 + |\nabla u|^2\big)\Delta u - \sum_{i,j=1}^{n} u_i u_j u_{ij}. \tag{1}$$

The graph of u is a minimal hypersurface of $\mathbb{R}^{n+1}$ exactly when $\mathcal{M}(u) = 0$. Indeed, with $W = \sqrt{1+|\nabla u|^2}$,

$$\operatorname{div} \frac{\nabla u}{W} = \frac{\mathcal{M}(u)}{W^3}, \tag{2}$$

and $W > 0$ everywhere. For polynomials, vanishing of $\mathcal{M}(u)$ as a polynomial and vanishing at every point of $\mathbb{R}^n$ are equivalent. We also write

$$\mathcal{L}(F) = |\nabla F|^2 \Delta F - \nabla F^{\mathsf{T}} (D^2 F) \nabla F, \qquad \text{so that} \qquad \mathcal{M}(u) = \Delta u + \mathcal{L}(u).$$

Theorem 1.1 (Main theorem)**.** *Let $n \geq 0$ and let $u \in \mathbb{R}[x_1, \dots, x_n]$ have total degree at most four. If $\mathcal{M}(u) = 0$, then u has total degree at most one.*

The theorem concerns *polynomial* solutions of total degree at most four, in *every* dimension n. It says nothing about non-polynomial entire solutions, or about polynomial solutions of degree five or more. Constants and the zero polynomial count as affine.

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Key words and phrases. Minimal surface equation, Bernstein problem, polynomial solutions, Lean.

Proof submitted by JenW1N. Working proof exposition and source guide prepared with AI assistance from the accepted Lean submission. The formal source is the authority for every step; this prose is not itself kernel-verified.

The formal statement. The accepted proof establishes the following proposition, quoted verbatim from the challenge's definition file (documentation comments omitted). Here `pderiv i` is the formal partial derivative and `totalDegree` is Mathlib's total degree, which is 0 on the zero polynomial.

```
abbrev RealPolynomial (n : ℕ) := MvPolynomial (Fin n) ℝ

noncomputable def minimalGraphOperator {n : ℕ} (P : RealPolynomial n) :
    RealPolynomial n :=
  (1 + ∑ i : Fin n, (MvPolynomial.pderiv i P) ^ 2) *
      (∑ i : Fin n, MvPolynomial.pderiv i (MvPolynomial.pderiv i P)) -
    ∑ i : Fin n, ∑ j : Fin n,
      MvPolynomial.pderiv i P * MvPolynomial.pderiv j P *
        MvPolynomial.pderiv i (MvPolynomial.pderiv j P)

def SatisfiesMinimalGraphEquation {n : ℕ} (P : RealPolynomial n) : Prop :=
  minimalGraphOperator P = 0

def IsAffine {n : ℕ} (P : RealPolynomial n) : Prop := P.totalDegree ≤ 1

def Problem28 : Prop :=
  ∀ (n : ℕ) (P : MinimalGraphs.RealPolynomial n),
    P.totalDegree ≤ 4 → MinimalGraphs.SatisfiesMinimalGraphEquation P →
      MinimalGraphs.IsAffine P

theorem target : fcTypeOfName% "Math30Catalog.source28"
```

The first three definitions live in the namespace `MinimalGraphs`. The operator is exactly (1), as a polynomial identity in the non-divergence form, so the formal statement is Theorem 1.1. The case $n = 0$, with no variables, is included and trivial.

1.1. **Context.** The question belongs to the Bernstein problem. Bernstein proved that entire solutions of the minimal surface equation on $\mathbb{R}^2$ are affine [1]; through the work of Fleming, De Giorgi, Almgren and Simons the same conclusion holds for all entire solutions on $\mathbb{R}^n$ with $n \leq 7$ [2, 3, 4, 5]. Bombieri, De Giorgi and Giusti constructed non-affine entire solutions on $\mathbb{R}^n$ for every $n \geq 8$ [6]; these are not polynomials. Whether a non-affine *polynomial* can solve the equation is the Bernstein-type problem for polynomial minimal graphs studied in [7].

Accordingly, for $n \leq 7$ Theorem 1.1 is a special case of the Bernstein theorem. Guo [7, Theorems 1.2 and 1.4] proves that a non-affine polynomial solution has degree at least four, in every dimension. The remaining content of Theorem 1.1 is therefore the exclusion of quartic polynomial solutions when $n \geq 8$. Guo's structure theorem [7, Theorem 1.3] also describes the top homogeneous component of a polynomial solution; the formal proof does not use it, and gives its own short arguments for degrees two and three. Our literature check was bounded and we make no priority claim.

The same solver's later submission for polynomial solutions on $\mathbb{R}^8$ of arbitrary degree [9] reuses the quartic, cubic and flux components described below.

1.2. **Structure of the proof.** Theorem 1.1 follows by cases on the degree. Degrees at most one are the conclusion. Degree two is Proposition 6.1, degree three is Proposition 6.3, and degree four is Theorem 5.1, which combines three ingredients:

(1) the top component $F = u_4$ satisfies $\mathcal{L}(F) = 0$ (Lemma 2.1);

(2) the top component of any non-affine solution, of any degree, takes both signs (Proposition 3.1);
(3) a homogeneous quartic with $\mathcal{L}(F) = 0$ has one weak sign (Theorem 4.1).

In Lean, the final declaration applies `affine_of_degree_le_four_minimal`, which performs this case split.

2. Preliminaries

Write $u = \sum_{j=0}^{d} u_j$ with u_j homogeneous of degree j and $u_d \neq 0$.

Lemma 2.1 (Leading equation)**.** *If $d \geq 2$ and $\mathcal{M}(u) = 0$, then $\mathcal{L}(u_d) = 0$.*

Proof. Every summand of $\mathcal{L}(u)$ has degree at most $3d-4$, and its component of degree $3d-4$ is $\mathcal{L}(u_d)$: it only receives contributions in which all three derivative factors fall on u_d. The Laplacian Δu has degree at most $d-2 < 3d-4$. Take the degree-$(3d-4)$ component of $\mathcal{M}(u) = \Delta u + \mathcal{L}(u) = 0$. □

The equation $\mathcal{L}(F) = 0$ alone is much weaker than minimality: $\mathcal{L}(\ell^k) = 0$ for every linear form ℓ and every $k \geq 1$ (Example 7.2). Information from the full equation is still needed; in degree four it enters through Proposition 3.1.

Lemma 2.2 (Orthogonal invariance)**.** *If U is an orthogonal matrix, then $\mathcal{M}(u \circ U) = \mathcal{M}(u) \circ U$ and $\mathcal{L}(u \circ U) = \mathcal{L}(u) \circ U$.*

Proof. The gradient and Hessian transform as $U^{\mathsf{T}} \nabla u(Ux)$ and $U^{\mathsf{T}} D^2 u(Ux)\, U$. The norm, trace and contraction in $\mathcal{M}$ and $\mathcal{L}$ are invariant under this conjugation. □

Lemma 2.3 (Normal form at a sphere extremum)**.** *Let F be homogeneous of degree $d \geq 1$, taking a positive value. Let e be a maximum point of F on the unit sphere, and $a = F(e) > 0$. For $y \perp e$,*

$$F(te + y) = at^d + \sum_{j=0}^{d-2} t^j F_j(y),$$

with F_j homogeneous of degree $d - j$; there is no t^{d-1} term. The same holds at a minimum point with a negative value.

Proof. The maximum exists by compactness, and is positive by homogeneity. By the Lagrange condition, $\nabla F(e) = \lambda e$; Euler's identity $e \cdot \nabla F(e) = dF(e)$ gives $\lambda = da$. The coefficient of t^{d-1} is the linear form $y \mapsto \nabla F(e) \cdot y = da\,(e \cdot y)$, which vanishes on $e^{\perp}$. □

By Lemma 2.2 we may choose orthonormal coordinates with e as the first axis, and work with $(t, y) \in \mathbb{R} \times \mathbb{R}^{n-1}$.

3. The top component changes sign

Proposition 3.1 (Flux obstruction)**.** *Let u be a polynomial of degree $d \geq 2$ with $\mathcal{M}(u) = 0$. Then u_d takes both a positive and a negative value.*

Proof. Since $\mathcal{M}(-u) = -\mathcal{M}(u)$, it suffices to show that $F = u_d$ is not everywhere nonnegative. Suppose $F \geq 0$.

Bounded divergence-free fields. Let $V = \nabla u / \sqrt{1 + |\nabla u|^2}$. It is C^1, each component satisfies $|V_i| \leq 1$, and $\operatorname{div} V = 0$ by (2). For $r_k = k + 1$ set $V_k(x) = V(r_k x)$. Then $\operatorname{div} V_k(x) = r_k (\operatorname{div} V)(r_k x) = 0$. For every compactly supported C^1 function φ, integration by parts gives

$$\int_{\mathbb{R}^n} \nabla \varphi \cdot V_k \, dx = -\int_{\mathbb{R}^n} \varphi \operatorname{div} V_k \, dx = 0. \tag{3}$$

The pointwise limit. Each u_i has degree at most $d-1$, with top component $\partial_i F$, so $r^{-(d-1)} \nabla u(rx) \to \nabla F(x)$ as $r \to \infty$. The zero set of the nonzero polynomial F has

Lebesgue measure zero. For almost every x we therefore have $F(x) > 0$, and Euler's identity $x \cdot \nabla F(x) = dF(x) > 0$ gives $\nabla F(x) \neq 0$. At such x, multiplying numerator and denominator of $V_k(x)$ by $r_k^{-(d-1)}$ shows

$$V_k(x) \longrightarrow G(x) := \frac{\nabla F(x)}{|\nabla F(x)|}.$$

Since $|\nabla\varphi \cdot V_k| \leq \sum_i |\partial_i \varphi|$, an integrable bound, dominated convergence in (3) gives $\int \nabla\varphi \cdot G\, dx = 0$. The limit G need not be continuous; only measurability and the uniform bound are used.

A radial test function. Let $\psi(s) = e^{-1/s}$ for $s > 0$ and $\psi(s) = 0$ for $s \leq 0$, and set $\varphi(x) = -\psi(1 - |x|^2)$. This function is smooth and supported in the closed unit ball, and

$$\nabla\varphi(x) = w(x)\, x, \qquad w(x) = \frac{2\,\psi(1-|x|^2)}{(1-|x|^2)^2} \geq 0,$$

with $w > 0$ on the open unit ball. Hence, for almost every x,

$$\nabla\varphi(x) \cdot G(x) = w(x) \frac{x \cdot \nabla F(x)}{|\nabla F(x)|} = w(x) \frac{d\,F(x)}{|\nabla F(x)|} \geq 0,$$

and this quantity is positive almost everywhere on the open unit ball, a set of positive measure. So $\int \nabla\varphi \cdot G\, dx > 0$, a contradiction. □

The argument uses neither $d = 4$ nor any restriction on n. In the formal proof it is `FluxPolynomial.top_changes_sign`, built from `homogeneous_nonnegative_limit_obstruction` and the general limit statement `integral_testFlux_limit_eq_zero`.

4. Homogeneous quartic solutions have one sign

Theorem 4.1 (Quartic sign theorem)**.** *Let F be a homogeneous real quartic in n variables with $\mathcal{L}(F) = 0$. Then $F \geq 0$ everywhere or $F \leq 0$ everywhere.*

The proof occupies this section. It is `homogeneous_quartic_has_one_sign` in the source.

4.1. **Expansion at a positive maximum.** Suppose F takes a positive value, and normalize at a maximum point e with $a = F(e) > 0$ as in Lemma 2.3:

$$F(te + y) = at^4 + t^2 A(y) + tB(y) + D(y), \qquad y \perp e,$$

where A, B, D are homogeneous of degrees $2, 3, 4$. All gradients, Hessians and Laplacians of A, B, D are taken in the variable y. Put

$$s = \partial_t F = 4at^3 + 2tA + B, \quad \mathbf{u} = \nabla_y \partial_t F = 2t\nabla A + \nabla B, \quad \mathbf{v} = \nabla_y F = t^2 \nabla A + t\nabla B + \nabla D.$$

Then $|\nabla F|^2 = s^2 + |\mathbf{v}|^2$, $\Delta F = 12at^2 + 2A + t^2\Delta A + t\Delta B + \Delta D$, and $\nabla F^{\mathsf{T}} D^2 F \nabla F = s^2 \partial_t^2 F + 2s\, \mathbf{u} \cdot \mathbf{v} + \mathbf{v}^{\mathsf{T}} D_y^2 F\, \mathbf{v}$. Substituting,

$$\mathcal{L}(F) = (12at^2 + 2A)|\mathbf{v}|^2 + \big(s^2 + |\mathbf{v}|^2\big)\big(t^2\Delta A + t\Delta B + \Delta D\big) - 2s\, \mathbf{u} \cdot \mathbf{v} - \mathbf{v}^{\mathsf{T}}\big(t^2 D^2 A + tD^2 B + D^2 D\big)\mathbf{v}. \quad (4)$$

Lemma 4.2 (Coefficient relations)**.** *If $\mathcal{L}(F) = 0$, then $\Delta A = \Delta B = 0$ and*

$$16a^2 \Delta D = 4a|\nabla A|^2 + \nabla A^{\mathsf{T}} (D^2 A) \nabla A. \quad (5)$$

Proof. View (4) as a polynomial in t. Since $s^2 = 16a^2 t^6 + \cdots$, the coefficients of t^8 and t^7 are $16a^2\Delta A$ and $16a^2 \Delta B$. With $\Delta A = 0$, the coefficient of t^6 is

$$16a^2\Delta D + 12a|\nabla A|^2 - 16a|\nabla A|^2 - \nabla A^{\mathsf{T}}(D^2 A)\nabla A,$$

where the terms $12a|\nabla A|^2$, $-16a|\nabla A|^2$ and $-\nabla A^{\mathsf{T}}(D^2 A)\nabla A$ come from the first, third and fourth terms of (4). All three coefficients vanish. □

The source treats the Laplacians as independent symbols h, k, l in a polynomial in t, and reads off the coefficients of t^8, t^7, t^6 (`HomogeneousQuarticCoefficients.operator_coeff6` and its neighbours).

4.2. **A Laplacian identity.** Write $A(y) = y^{\mathsf{T}} S y$ with S symmetric on $e^{\perp}$. Then $\nabla A = 2Sy$ and $D^2 A = 2S$, so (5) reads $\Delta D(y) = y^{\mathsf{T}}\big(a^{-1}S^2 + (2a^2)^{-1}S^3\big)y$. Together with $\Delta A = \Delta B = 0$ this gives

$$\Delta F(te + y) = 12at^2 + y^{\mathsf{T}} \mathcal{F}_a(S)\, y, \qquad \mathcal{F}_a(S) = 2S + a^{-1}S^2 + (2a^2)^{-1}S^3. \tag{6}$$

Since ΔF is a quadratic form, $\Delta F(x) = x^{\mathsf{T}} Q x$ for a unique symmetric matrix Q, which depends only on F. Extend S to $\mathbb{R}^n$ by $Se = 0$. At $x = e$ the Hessian of F is $12a\, ee^{\mathsf{T}} + 2S$, since ∇A and ∇B vanish at $y = 0$. Thus, intrinsically,

$$S = \tfrac{1}{2} D^2 F(e) - 6a\, ee^{\mathsf{T}}, \tag{7}$$

and (6) becomes the matrix identity $Q = 12a\, ee^{\mathsf{T}} + \mathcal{F}_a(S)$. In particular

$$Qe = 12a\, e, \qquad Qx = \mathcal{F}_a(S)\, x \quad \text{for } x \perp e. \tag{8}$$

In the source, (6) is `homogeneous_quartic_laplacian_at`, (7) is `canonicalS`, and (8) is `ambient_Q_action`.

4.3. **A sign lemma.** For real λ one has $2\lambda + a^{-1}\lambda^2 + (2a^2)^{-1}\lambda^3 = \frac{\lambda}{2}\big((\lambda/a + 1)^2 + 3\big)$, which has the sign of λ. The source uses the following matrix version, proved by a sum of squares rather than by diagonalization.

Lemma 4.3. *Let $a \neq 0$, let S be symmetric, and suppose $\mathcal{F}_a(S)v = cv$ with $v \neq 0$ and $c \neq 0$. Then $c\, v^{\mathsf{T}} S v > 0$.*

Proof. Expanding and using the symmetry of S,

$$(Sv) \cdot \mathcal{F}_a(S)v = \tfrac{3}{2}|Sv|^2 + \frac{1}{2a^2}|S^2 v + aSv|^2.$$

If $Sv = 0$ then $\mathcal{F}_a(S)v = 0 \neq cv$; so $Sv \neq 0$ and the right side is positive. The left side equals $(Sv) \cdot cv = c\, v^{\mathsf{T}} S v$. □

4.4. **Proof of Theorem 4.1.** For $n = 0$ there is nothing to prove. Suppose F takes both signs. Let e be a maximum point on the unit sphere with $a = F(e) > 0$, and f a minimum point with $b = F(f) < 0$. Define S by (7) and, likewise, $T = \frac{1}{2} D^2 F(f) - 6b\, ff^{\mathsf{T}}$. Applying the preceding subsections at both points, with the same matrix Q,

$$Qe = 12a\, e, \qquad Qf = 12b\, f.$$

The eigenvalues $12a \neq 12b$ are distinct, so $e \perp f$. Then (8) at e and its analogue at f give

$$\mathcal{F}_a(S) f = 12b\, f, \qquad \mathcal{F}_b(T) e = 12a\, e.$$

Lemma 4.3 yields $f^{\mathsf{T}} S f < 0$ and $e^{\mathsf{T}} T e > 0$. But $e \perp f$ removes the rank-one terms, so

$$f^{\mathsf{T}} S f = \tfrac{1}{2} f^{\mathsf{T}} D^2 F(e) f, \qquad e^{\mathsf{T}} T e = \tfrac{1}{2} e^{\mathsf{T}} D^2 F(f) e,$$

and for a homogeneous quartic $f^{\mathsf{T}} D^2 F(e) f = e^{\mathsf{T}} D^2 F(f) e$: both equal $12\,\mathcal{T}(e, e, f, f)$, where $\mathcal{T}$ is the symmetric four-linear form with $F(x) = \mathcal{T}(x, x, x, x)$. This contradiction proves the theorem. □

The contradiction step is `two_critical_directions_from_laplacian`; the mixed symmetry is `quartic_mixed_hessian_symmetry` and `canonicalS_mixed_symmetry`.

5. Degree four

Theorem 5.1. *No real polynomial of total degree exactly four satisfies $\mathcal{M}(u) = 0$.*

Proof. By Lemma 2.1, $F = u_4$ satisfies $\mathcal{L}(F) = 0$, so F has one weak sign by Theorem 4.1. By Proposition 3.1, F takes both signs. □

This is `no_degree_four_minimal_polynomial`. Neither ingredient suffices alone: Example 7.2 gives one-signed quartics with $\mathcal{L}(F) = 0$, and Proposition 3.1 holds in every degree.

6. Degrees two and three

Both cases are known [7]. The formal proof includes the following elementary arguments.

Proposition 6.1 (Quadratic case)**.** *If $\deg u \le 2$ and $\mathcal{M}(u) = 0$, then u is affine.*

Proof. Write $\nabla u(x) = Hx + c$ with H symmetric and constant. By Lemma 2.2 we may take $H = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$. With $v_i = \lambda_i x_i + c_i$,

$$\mathcal{M}(u)(x) = \Big(1 + \sum_i v_i^2\Big) \sum_j \lambda_j - \sum_i \lambda_i v_i^2.$$

Choose $x_i = -c_i/\lambda_i$ whenever $\lambda_i \neq 0$. Every term $\lambda_i v_i^2$ vanishes, so $\operatorname{tr} H = \sum_j \lambda_j = 0$. If some $\lambda_j \neq 0$, change x_j so that $v_j = 1$; then $\mathcal{M}(u)(x) = -\lambda_j \neq 0$. Hence $H = 0$. □

Lemma 6.2 (Homogeneous cubics)**.** *If F is a nonzero homogeneous cubic with $\mathcal{L}(F) = 0$, then in suitable orthonormal coordinates $F = at^3$ with $a > 0$.*

Proof. An odd nonzero polynomial takes a positive value. Normalize at a sphere maximum: $F(te + y) = at^3 + tA(y) + B(y)$, with A, B of degrees two and three. With $s = 3at^2 + A$ and $\mathbf{v} = t\nabla A + \nabla B$,

$$\mathcal{L}(F) = 6at|\mathbf{v}|^2 + \big(s^2 + |\mathbf{v}|^2\big)(t\Delta A + \Delta B) - 2s\nabla A \cdot \mathbf{v} - \mathbf{v}^{\mathsf{T}}(tD^2A + D^2B)\mathbf{v}.$$

The coefficients of t^5 and t^4 are $9a^2\Delta A$ and $9a^2\Delta B$. With these zero, the coefficient of t^3 is $-\nabla A^{\mathsf{T}}(D^2A)\nabla A = -8\, y^{\mathsf{T}} S^3 y$ for $A = y^{\mathsf{T}} S y$. So $S^3 = 0$, hence $S = 0$ for symmetric S, and $A = 0$. Then the coefficient of t is $6a|\nabla B|^2$, so $\nabla B = 0$ and $B = 0$. □

Proposition 6.3 (Cubic case)**.** *No real polynomial of total degree exactly three satisfies $\mathcal{M}(u) = 0$.*

Proof. By Lemmas 2.1, 2.2 and 6.2 we may assume $u_3 = at^3$ with $a > 0$. Since $u - u_3$ has degree at most two,

$$u = at^3 + bt^2 + t(\alpha \cdot y + c) + B(y), \qquad \nabla B(y) = Ky + \delta,$$

with constants b, c, vectors α, δ and a symmetric matrix K. As a polynomial in t, $\mathcal{M}(u)$ has degree at most four, with t^4 coefficient $9a^2 \operatorname{tr} K$. So $\operatorname{tr} K = 0$, and then the coefficient of t is

$$h(y) = 6a\big(1 + |Ky + \delta|^2\big) - 2(c + \alpha \cdot y)|\alpha|^2 - 2\,\alpha^{\mathsf{T}} K(Ky + \delta);$$

the terms in b cancel. Now $h(y) + h(-y) - 2h(0) = 12a|Ky|^2$, so $K = 0$. Then $h(\alpha) - h(0) = -2|\alpha|^4$ forces $\alpha = 0$, and finally $h(0) = 6a(1 + |\delta|^2) \neq 0$, a contradiction. □

7. Worked examples

The following identities are verified symbolically by the accompanying `validate.py`.

Example 7.1. The harmonic quadratic $u = x_1^2 - x_2^2$ has $\mathcal{M}(u) = 8(x_2^2 - x_1^2) \neq 0$. Harmonicity is not enough; in the proof of Proposition 6.1, $\operatorname{tr} H = 0$ but $\lambda_1 = 2 \neq 0$.

Example 7.2. For $u = x_1^4$ one has $\mathcal{L}(u) = 0$ and $\mathcal{M}(u) = 12x_1^2 \neq 0$. More generally, for $\ell = x_1 + 2x_2 - 2x_3$, $\mathcal{L}(\ell^3) = \mathcal{L}(\ell^4) = 0$ and $\mathcal{M}(\ell^4) = 108\,\ell^2$. These quartics satisfy the leading equation and have one sign, as Theorem 4.1 permits; they are excluded as top components by Proposition 3.1. By contrast the sign-changing quartic $F = x_1^4 - x_2^4$ has $\mathcal{L}(F) = -192\, x_1^2 x_2^2 F \neq 0$.

The script also checks, with generic symbolic coefficients, the divergence identity (2), the leading-component statement of Lemma 2.1, the expansion (4) and the coefficients in Lemma 4.2, the identity (6), the sum of squares in Lemma 4.3, the mixed Hessian symmetry, the cubic expansions in Lemma 6.2 and Proposition 6.3, and the radial test function. These are transcription checks. The theorem rests on the accepted Lean proof.

8. Formal verification and provenance

The submission was accepted by the service on 7 October 2026 at 19:28:16 UTC. Production verification run 42 completed at 19:32:03 UTC and returned `VERIFIED`. A human-authorized review approved it on 8 October 2026, under review policy v3, and the reward was paid on 9 October. The public proof was retrieved for this paper and checked byte for byte against the accepted artifact.

The run passed the statement comparator, task commitments, trusted-file checks, permitted-axiom check and Lean kernel check under `landrun+seccomp`. The allowed axioms were `propext`, `Quot.sound`, and `Classical.choice`. Nanoda was disabled. The verifier image pins Lean toolchain `v4.35.0-rc2`, a locally patched Formal Conjectures revision `4b69a7dca3e7` based on upstream `7d1a8c991274`, and Mathlib revision `065356127b1d`. No new Lean replay was performed for this paper. The pre-approval review covered the exact definitions and selected proof interfaces; it was not a line-by-line audit.

The submitted file adapts a few Mathlib proofs (polynomial extensionality, evaluation compatibility, and Euler's identity for homogeneous polynomials) with their Apache 2.0 notices and authors retained. JenW1N is the public solver name attached to the submitted proof [8].

9. Guide to the complete source

Line numbers below refer to the exact public submitted `Main.lean`, which has 4,947 lines; lines 4742–4947 are the Apache license text. They count submitted bytes only, without the validator's trusted header and footer. Names are given relative to the enclosing namespace. The source checksum is

`805bf4ea30894ebf417d190e6712fa76609`
`593cb0cbed5a1e42c38e475fdddcb`.

Declaration and starting line	Mathematical role
`minimalGraphOperator` (399), `cubicOperator` (437)	The operator $\mathcal{M}$ and its cubic part $\mathcal{L}$.
`homogeneousComponent_minimalGraphOperator_top` (488), `cubicOperator_top_eq_zero` (503)	Lemma 2.1.
`minimalOperator` (903), `leadingOperator` (961)	Matrix-form copies of $\mathcal{M}$ and $\mathcal{L}$, definitionally equal.
`minimalOperator_changeVariables` (942)	Lemma 2.2.
`exists_gradient_multiplier_of_local_extremum_on_sphere` (618)	Lagrange condition in Lemma 2.3.
`QuarticNormalization.exists_positive_normal_form` (2704), `exists_negative_normal_form` (2718)	Lemma 2.3 at a maximum and a minimum.
`HomogeneousQuarticCoefficients.operator_coeff6` (1533)	The t^6 coefficient in Lemma 4.2.
`laplacian_polynomial_relation` (1635)	The matrix form of (6).
`QuarticNormalForm.homogeneous_quartic_constraints` (1832)	Lemma 4.2 for an actual quartic.
`homogeneous_quartic_laplacian_at` (1934)	Identity (6).
`AmbientQuartic.canonicalS` (2007), `ambient_Q_identity` (2273), `ambient_Q_action` (2336)	Equations (7)–(8).
`quartic_mixed_hessian_symmetry` (1385), `canonicalS_mixed_symmetry` (2364)	$f^{\mathsf{T}}Sf = e^{\mathsf{T}}Te$ when $e \perp f$.

Declaration and starting line	Mathematical role
`QuarticSpectral.scalarFa_factor` (2750), `matrixFa_energy` (2804), `eigenvalue_times_quadratic_pos` (2820)	Lemma 4.3.
`two_critical_directions_from_laplacian` (2879)	Final contradiction for Theorem 4.1.
`homogeneous_quartic_no_opposite_signs` (2912), `homogeneous_quartic_has_one_sign` (2948)	Theorem 4.1.
`QuadraticScalar.diagonal_quadratic_obstruction` (3045), `quadratic_matrix_obstruction` (3069), `affine_of_quadratic_minimal` (3124)	Proposition 6.1.
`CubicNormalForm.actual_constraints_of_normalForm` (3428), `CubicMatrix.symmetric_eq_zero_of_cube_quadratic_zero` (3330), `homogeneous_cubic_pure` (3490), `exists_orthogonal_pure_cubic` (3830)	Lemma 6.2.
`FullCubicCoefficients.operator_coeff1_of_trace_zero` (3590), `CubicScalar.obstruction` (3625), `normalForm_not_minimal` (3771), `no_degree_three_minimal_polynomial` (3844)	Proposition 6.3.
`FluxObstruction.integral_testFlux_limit_eq_zero` (4011), `testFlux_limit_integrable` (4037)	Flux identity and dominated convergence.
`radialTest` (4079), `divergence_normalizedField` (4250), `divergence_dilate` (4266)	Test function, identity (2), rescaling.
`normalized_ratio_tendsto` (4303), `mvPolynomial_eval_ne_zero_ae` (4341)	Pointwise limit; null zero sets.
`homogeneous_nonnegative_limit_obstruction` (4436)	The radial flux contradiction.
`FluxPolynomial.top_not_nonnegative` (4607), `top_changes_sign` (4674)	Proposition 3.1.
`no_degree_four_minimal_polynomial` (4704)	Theorem 5.1.
`affine_of_degree_le_four_minimal` (4717), `target` (4734)	Case split and the exact committed statement.

The formal source contains the complete development, including the coordinate bookkeeping for distinguished variables, the polynomial coefficient expansions, compactness and Lagrange multipliers on the sphere, and the measure theory of the flux argument. The prose above gives the mathematical content of each step and locates it in the table.

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