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WILF'S INEQUALITY FOR NUMERICAL SEMIGROUPS WITH CONDUCTOR AT MOST FOUR TIMES THE MULTIPLICITY

Abstract. For a numerical semigroup with conductor c, multiplicity m, embedding dimension e, and ℓ elements below its conductor, Wilf's inequality is $c \le e\ell$. We reconstruct a proof for $c \le 4m$ from the formal solution credited to Elias. The depth-four argument combines canonical monomial shadows, two anchor injections, a full strip-extension estimate, and a quantitative comparison with the semigroup generated by primitives below $2m$. Explicit polynomial certificates establish the coefficient $23/13$ in that comparison. A colored-shadow argument treats the complementary population case, giving an exhaustive depth-four proof. Together with the established depth-three theorem of Eliahou, this yields the asserted universal conductor range.

1. The assertion and its scope

A numerical semigroup is an additive submonoid $S \subseteq \mathbb{N}$ with finite complement. We use $0 \in \mathbb{N}$. For a proper numerical semigroup define

$$m = \min(S \setminus \{0\}), \qquad c = 1 + \max(\mathbb{N} \setminus S), \qquad L = S \cap [0, c), \quad \ell = |L|.$$

A positive element of S is *primitive* if it is not a sum of two positive elements of S. Let P be the set of primitive elements and put $e = |P|$. In particular 0 is counted in ℓ and is not counted in e.

The statement represented by the terminal declaration `target` in the submitted source is

$$\text{for every proper numerical semigroup } S, \qquad c \le 4m \implies c \le e\ell. \tag{1}$$

There is no bound on m, no divisibility assumption on c, and no restriction on e. If the convention admitting $S = \mathbb{N}$ is used, that exceptional case has $c = 0$ and satisfies the inequality immediately.

The submitted proof is credited to Elias. The exposition here is a reconstruction from its mathematical content, not an assertion that the submitter authored or approved this manuscript. The established $c \le 3m$ theorem of Eliahou is relevant prior work [2]. Its scope alone does not imply (1). The present manuscript concentrates on the additional depth-four machinery and does not make an exhaustive historical-priority claim.

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Key words and phrases. Numerical semigroups, Wilf's inequality, conductor, multiplicity, Lean.

About this exposition. The formal proof is credited to Elias; the explanatory manuscript was reconstructed from that submission and reviewed by a separate AI agent. The prose is not kernel checked, and no fresh Lean build was performed while preparing it. Its mathematical arguments and the formal source remain distinct artifacts.

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2. Basic semigroup counting

Lemma 2.1. *Every positive element of S is a finite sum of primitives. The set P is contained in $[m, c+m)$, and $m \in P$.*

Proof. Induct strongly on a positive element $x \in S$. If x is primitive there is nothing to prove. Otherwise $x = a + b$ for positive $a, b \in S$, each strictly below x, and the induction hypothesis applies. Every positive element is at least m. If $x \geq c + m$, then $x - m \geq c$ belongs to S and is positive; hence $x = m + (x - m)$ is not primitive. Finally a decomposition of m into two positive semigroup elements would have sum at least $2m > m$. □

Formal correspondence: `w22_sum_of_primitives; w22_primitives_subset`.

Formal correspondence: `w22_multiplicity_mem_primitives`.

Let $n = |(|P \cap [0, c))$ and $p = |(|P \cap [c, c+m))$. Every element of $[c, c+m)$ belongs to S. Write D for the decomposable elements of that interval and $d = |D|$. Lemma 2.1 gives

$$e = n + p, \qquad m = d + p. \tag{2}$$

Let $q = \lceil c/m \rceil$ and $\rho = qm - c$, so $0 \leq \rho < m$. Define

$$W_0 = n\ell - qd + \rho, \qquad W = e\ell - c. \tag{3}$$

Then the exact identity

$$W = W_0 + p(\ell - q) \tag{4}$$

follows by substituting (2) and $c = qm - \rho$. Also $0, m, \ldots, (q-1)m$ all belong to L, so $\ell \geq q$. Consequently $W_0 \geq 0$ implies Wilf's inequality.

Theorem 2.2. *If $c \leq 2m$, then $c \leq e\ell$.*

Proof. First $m \leq c$, since c is a positive semigroup element. If $c = m$, then $S = \{0\} \cup [m, \infty)$, $P = [m, 2m)$, and $\ell = 1$, $e = m$, giving equality.

Suppose $m < c \leq 2m$. Every positive element below c is primitive: a sum of two positive elements is at least $2m$. Therefore $\ell = n + 1$ and $n \geq 1$. If $x \in D$, choose $x = a + b$ with a, b positive in S. Since $x < c + m$ and each summand is at least m, both summands are below c and hence primitive. The number of unordered pairs of left primitives, allowing repetition, is $n(n+1)/2$. Therefore

$$m = d + p \leq \frac{n(n+1)}{2} + p.$$

It follows that

$$e\ell = (n+p)(n+1) \geq n(n+1) + 2p \geq 2m \geq c,$$

where the first inequality uses $p(n-1) \geq 0$. □

Formal correspondence: `w22_block_coverage; w22_kaplan_count`.

Formal correspondence: `w22_kaplan`.

3. The depth-four level model

Throughout the remainder put $c = 4m - \rho$, where $0 \leq \rho < m$. The level of an integer $x \geq 0$ is $\lfloor (x + \rho)/m \rfloor$; its shifted position is $(x + \rho) \bmod m$. Thus level k consists of the integers in $[km - \rho, (k+1)m - \rho)$, restricted to $\mathbb{N}$.

Let $B \subseteq \{1, \ldots, m-1\}$ and let $H = \{0\} \cup B$. Define

$$T = \langle m,\ m + b\ (b \in B) \rangle\ \cup\ [4m - \rho, \infty).$$

This is a numerical semigroup. It need not have conductor exactly $4m-\rho$ without a further condition. For $t \geq 0$, tH denotes the sums of exactly t elements of H; because $0 \in H$, these are equivalently sums of at most t offsets in B. Put

$$\mathcal{B}_k = \left\{ (s+\rho) \bmod m : s \in tH, \quad t + \left\lfloor \frac{s+\rho}{m} \right\rfloor \leq k \text{ for some } t \geq 0 \right\}. \tag{5}$$

Only $t \leq k$ can occur, so these are finite sets.

Lemma 3.1. *For $u \in [0,m)$, $u \in \mathcal{B}_k$ if and only if*

$$km + u - \rho \in \langle m, m+b : b \in B \rangle.$$

In this assertion membership on the right requires the displayed integer to be nonnegative.

Proof. A sum of t generators is $tm+s$ with $s \in tH$. Write $s+\rho = jm+u$. Its level is $t+j$. If $t+j \leq k$, add $k-t-j$ copies of m, obtaining $km+u-\rho$. Conversely, if $km+u-\rho = tm+s$ with $s \in tH$, then $s+\rho = (k-t)m+u$. Nonnegativity and $u<m$ force $t \leq k$. The quotient and remainder are $k-t$ and u, so (5) holds. □

Formal correspondence: `wm_B_spec; wm_B_of_sum`.

Suppose the conductor of T is exactly $c = 4m-\rho$. Its multiplicity is m, and all its left primitives are precisely m and $m+b$, $b \in B$. Indeed these generators lie in $[m,2m)$ and cannot decompose. Every other positive element below c is a sum of at least two generators. Hence

$$\ell = \sum_{k=0}^{3} |\mathcal{B}_k|, \qquad n = |B|+1, \quad d = |\mathcal{B}_4|, \quad p = m-d. \tag{6}$$

To verify the formula for d, an element of the conductor interval is decomposable exactly when it lies in the generated monoid. A decomposition there uses two positive elements below c, since either summand is at most $x-m<c$. Conversely a generated element in that interval is a sum of at least two generators, because every individual generator is below $2m<c$.

The conductor condition is also expressible entirely in the model:

$$c \text{ is the conductor of } T \quad \Longleftrightarrow \quad m-1 \notin \mathcal{B}_3. \tag{7}$$

Indeed $c-1 = 3m+(m-1)-\rho$, and all integers at least c were adjoined by definition. A missing $c-1$ is both necessary and sufficient for this threshold to be the least conductor.

Formal correspondence: `wm_of_core; wm_p_nonneg`.

Split the offsets into the *window* and *strip*:

$$O = \{b \in B : b+\rho < m\}, \qquad G = \{b \in B : m \leq b+\rho\}.$$

Write $N = m-\rho$. Window offsets satisfy $0<b<N$, whereas strip offsets satisfy $N \leq b < m$. The next sections treat $B = O$, with $G = \varnothing$.

4. Unwrapped and wrapped sums

For a window set $O \subseteq [1,N)$, define

$$Y_t = \{s+\rho : s \in t(\{0\} \cup O),\ s < N\},$$
$$Z_t = \{s+\rho-m : s \in t(\{0\} \cup O),\ N \leq s < N+m\}.$$

Both are subsets of $[0,m)$. They are increasing with t. We have $Y_0 = \{\rho\}$, $Y_1 = \rho + (\{0\} \cup O)$, and $Z_0 = Z_1 = \varnothing$.

Lemma 4.1. *For $0 \leq k \leq 4$,*

$$\mathcal{B}_k = Y_k \cup Z_{k-1},$$

where $Z_{-1} = \varnothing$. In particular $\mathcal{B}_k = Y_k$ for $k \leq 2$.

Proof. For a representation contributing to (5), write $j = \lfloor (s+\rho)/m \rfloor$. If $j = 0$, then $t \le k$, so padding with zero offsets shows that the position belongs to Y_k. If $j = 1$, then $t \le k-1$, giving membership in Z_{k-1}. If $j \ge 2$, then $t \le 2$. But a sum of at most two window offsets satisfies $s + \rho < 2(m-\rho) + \rho \le 2m$, a contradiction. Conversely either of the two stated memberships provides a representation satisfying (5). Finally a once-wrapped sum of at most one window offset is impossible. □

Formal correspondence: `wm_mem_B_free; wm_B_eq_Y`.

Formal correspondence: `wm_B_eq_YZ`.

Put $K = |O|$ and define

$$\widehat{Z} = Z_2 \setminus Y_3, \quad Q = Z_3 \setminus (Y_4 \cup Z_2), \quad N_{40} = (Y_4 \setminus Y_3) \setminus Z_2.$$

Then

$$\mathcal{B}_3 = Y_3 \sqcup \widehat{Z}, \tag{8}$$

$$\mathcal{B}_4 = Y_3 \sqcup \widehat{Z} \sqcup N_{40} \sqcup Q. \tag{9}$$

These unions are disjoint by construction and Lemma 4.1. Let

$$h_t = |Y_t| - |Y_{t-1}| \quad (t \ge 1), \qquad z = |\widehat{Z}|, \quad \eta = |\widehat{Z} \cap Y_4|.$$

In particular $h_1 = K$ and $|N_{40}| \le h_4$. By (8)–(9),

$$\ell = 4 + 3K + 2h_2 + h_3 + z, \tag{10}$$

$$d = K + 1 + h_2 + h_3 + h_4 + z - \eta + |Q|. \tag{11}$$

These quantities, and $W_0 = (K+1)\ell - 4d + \rho$, may be used for every window model, even before a conductor condition is imposed.

5. Canonical representations and weighted shadows

List the offsets in increasing order $a_0 < \cdots < a_{K-1}$. A monomial is an exponent vector $\alpha \in \mathbb{N}^K$, with degree $\deg \alpha = \sum_i \alpha_i$ and value $v(\alpha) = \sum_i \alpha_i a_i$. For a fixed degree $t \le 4$ and a representable value s, choose the representation minimizing

$$C(\alpha) = \sum_{i=0}^{K-1} \alpha_i 6^i.$$

The minimum exists because there are finitely many degree-t vectors. It is unique: every coordinate is less than 6, so base-6 expansion is unique. Denote it by $\mathrm{rep}_t(s)$.

Lemma 5.1. *If $\alpha = \mathrm{rep}_t(s)$ and $\alpha_i > 0$, then*

$$\alpha - \mathbf{e}_i = \mathrm{rep}_{t-1}(s - a_i).$$

Proof. The left side is a representation of the asserted degree and value. Any representation γ of the same degree and value with a smaller code would give $\gamma + \mathbf{e}_i$ of degree t, value s, and code smaller by the same amount than $C(\alpha)$. This contradicts minimality. □

Formal correspondence: `wm_rep_spec; wm_rep_parent`.

Let M_t consist of the canonical degree-t vectors with values s such that $s + \rho \in Y_t \setminus Y_{t-1}$. Such a new value necessarily needs exactly t nonzero offsets; otherwise it already lies in Y_{t-1}. Thus

$$|M_t| = h_t. \tag{12}$$

The value determines the position and the canonical vector determines its value, so this equality counts distinct positions, without confusing multiplicities of representations.

Lemma 5.2. *If $\alpha \in M_t$, $2 \le t \le 4$, and $\alpha_i > 0$, then $\alpha - \mathbf{e}_i \in M_{t-1}$.*

Proof. The parent has smaller value, so it remains unwrapped. Lemma 5.1 proves it is canonical. If its value had a representation using at most $t-2$ offsets, adding a_i would give a representation of $v(\alpha)$ using at most $t-1$ offsets, contrary to its being new in degree t. □

Formal correspondence: `wm_M_card; wm_M_parent`.

Lemma 5.3 (Weighted shadow count)**.** *Let J be a set of l coordinates. Suppose every vector of a finite set E has total degree r in J, and every vector of a finite set F has degree $r+1$ in J. Suppose each parent $\beta - \mathbf{e}_i$, for $\beta \in F$, $i \in J$, $\beta_i > 0$, lies in E. Then*

$$(r+1)|F| \le (l+r)|E|.$$

Proof. For a fixed i, the map $\beta \mapsto \beta - \mathbf{e}_i$ is injective on the vectors with $\beta_i > 0$. A parent α receives weight $\beta_i = \alpha_i + 1$. Thus

$$\sum_{\beta \in F} \beta_i \le \sum_{\alpha \in E} (\alpha_i + 1).$$

Sum over $i \in J$. The left side becomes $(r+1)|F|$ and the right side $(r+l)|E|$. □

Formal correspondence: `gm_weighted_shadow`.

6. The nonnegative collision and cubic terms

A pair of offsets always means an unordered pair with repetition. Let P_u be the number of pairs $a \le b$ with $a+b+\rho < m$, and P_w the number with $a+b+\rho \ge m$. Then

$$P_u + P_w = \frac{K(K+1)}{2}.$$

Every new unwrapped degree-two value is represented by an unwrapped pair. Every element of $\widehat{Z}$ is represented by a wrapped pair: a wrapped sum using at most two offsets cannot use zero or one offset. Hence

$$h_2 \le P_u, \qquad z \le P_w.$$

The *collision term*

$$\mathrm{coll} = 6(P_u - h_2) + 6(P_w - z) \tag{13}$$

is nonnegative. The inequalities allow several formal pairs to have the same value; those collisions are precisely why the differences must be retained.

Formal correspondence: `wm_Pu_ge; wm_Pw_ge`.

Formal correspondence: `wm_coll_nonneg`.

Every $q \in Q$ is a once-wrapped value requiring exactly three offsets. Put

$$\beta_q = \mathrm{rep}_3(q+m-\rho), \qquad w(q) = \sum_{a_i \le q} \beta_{q,i}, \qquad Q_j = \{q \in Q : w(q) = j\} \quad (0 \le j \le 3).$$

The weight counts factors whose removal leaves a wrapped pair. Indeed the parent value is $q + m - \rho - a_i$, which is wrapped exactly when $a_i \le q$.

For $z \in \widehat{Z}$ put $\kappa_z = \mathrm{rep}_2(z+m-\rho)$. If $\beta_{q,i} > 0$ and $a_i \le q$, then the wrapped parent is κ_{q-a_i} and $q - a_i \in \widehat{Z}$. The parent is canonical by Lemma 5.1. If its position were in Y_3, adding a_i would put q in Y_4, contradicting the definition of Q.

If instead $a_i > q$, that parent lies in M_2. It is unwrapped; if its value used at most one offset, the original cubic value would use at most two offsets and q would lie in Z_2, again impossible.

Define

$$\delta = (K+2)h_2 - 3h_3 - \sum_{q \in Q} (3 - w(q)). \tag{14}$$

Lemma 6.1. $\delta \ge 0$.

Proof. Fix i. Consider both the vectors $\beta \in M_3$ with $\beta_i > 0$ and the vectors β_q for $q \in Q$ with $\beta_{q,i} > 0$ and $a_i > q$. The two families are disjoint because the first has unwrapped value and the second has wrapped value. The parent map in direction i is injective on their union and lands in M_2. It follows that

$$\sum_{\beta \in M_3} \beta_i + \sum_{\substack{q \in Q \\ a_i > q}} \beta_{q,i} \le \sum_{\alpha \in M_2} (\alpha_i + 1).$$

Sum over i. On the left the first term is $3h_3$ and the second is $\sum_q (3 - w(q))$. On the right it is $(K+2)h_2$. This is exactly (14). □

Formal correspondence: `wm_beta_parent_unwrapped; wm_delta_coord`.

Formal correspondence: `wm_d3pp_nonneg`.

7. The residual quartic inequality

Call $a \in O$ low if $2a + \rho < m$ and high if $2a + \rho \ge m$. Let l and h be the respective counts, so $K = l + h$. The formal residual hypotheses include

$$K \ge 4, \qquad h \ge 3, \qquad 3l(l+1) < K(K-1), \qquad |\{a \in O : 3a + \rho < 2m\}| > 1. \tag{15}$$

Only the middle two inequalities are needed for the quartic estimate.

Lemma 7.1. *If $h \ge 3$ and $3l(l+1) < (l+h)(l+h-1)$, then*

$$2l + 1 \le 3h, \qquad l + 3 \le 4h.$$

Proof. If $2l + 1 > 3h$, integrality gives $2l \ge 3h$. Therefore

$$\begin{aligned}(l+h)(l+h-1) - 3l(l+1) &= -2l^2 + 2lh + h^2 - 4l - h \\ &\le -lh + h^2 - 4l - h < 0,\end{aligned}$$

since $l \ge 3h/2$ and $h > 0$. This contradicts the hypothesis. The second inequality follows from $2l \le 3h - 1$ and $h \ge 3$. □

Theorem 7.2. *Under these residual hypotheses, if*

$$d_4 = (K+3)h_3 - 4h_4,$$

then $d_4 \ge h_4$.

Proof. An unwrapped vector contains at most one high factor, counted with multiplicity. Two high factors have sum at least $m - \rho$, which would make the value wrapped. Partition $M_t = M_{t,0} \sqcup M_{t,1}$ according to whether there are zero or one high factors. Apply Lemma 5.3 using only the l low coordinates. Removing a low factor preserves the high count and remains in M_{t-1} by Lemma 5.2. It follows that

$$4|M_{4,0}| \le (l+3)|M_{3,0}|, \qquad 3|M_{4,1}| \le (l+2)|M_{3,1}|.$$

Lemma 7.1 implies

$$5(l+3) \le 4(K+3), \qquad 5(l+2) \le 3(K+3).$$

Multiplying the two shadow inequalities by $5/4$ and $5/3$ respectively gives

$$5|M_{4,j}| \le (K+3)|M_{3,j}| \quad (j = 0, 1).$$

Their sum says $5h_4 \le (K+3)h_3$, which is the desired assertion. □

Formal correspondence: `wm_res_arith; wm_quartic`.

8. An exact accounting identity

A pair-direction incidence (z, i) with $z \in \widehat{Z}$ is *used* if

$$z + a_i \in Q \quad \text{and} \quad \beta_{z+a_i} = \kappa_z + \mathbf{e}_i.$$

Give it weight $\kappa_{z,i} + 1$, and let A be the total weight of unused incidences. The total weight at a fixed z is $K + 2$. Each used incidence corresponds uniquely to a wrapped parent of a canonical cubic, with weight equal to that factor's multiplicity. Consequently

$$A = (K + 2)z - \sum_{q \in Q} w(q). \tag{16}$$

Define

$$\Sigma = |Q_1| + A + 2|Q_0| + \mathrm{coll} + 4\eta + 2\delta + d_4.$$

Proposition 8.1. *For every strip-free model,*

$$W_0 = \rho + z + \Sigma - |Q_3|. \tag{17}$$

Proof. Insert (10) and (11) into $W_0 = (K + 1)\ell - 4d + \rho$. This yields

$$\begin{aligned} W_0 = \rho &+ 3K(K + 1) + (2K - 2)h_2 + (K - 3)h_3 - 4h_4 \\ &+ (K - 3)z + 4\eta - 4|Q|. \end{aligned}$$

On the other hand substitute (13), (14), (16) into Σ, and use

$$\sum_{q \in Q} w(q) = |Q_1| + 2|Q_2| + 3|Q_3|, \qquad |Q| = \sum_{j=0}^{3} |Q_j|.$$

The result differs from the displayed formula for W_0 by exactly $-\rho - z + |Q_3|$, proving (17). □

Formal correspondence: `wm_ID2`.

9. Anchors and the two injections

Assume there is at least one high offset. For a high c let

$$r(c) = |\{a \in O : a \text{ high},\ a < c\}|, \qquad D_c = \{a \in O : a + c < N\}.$$

Choose a high y minimizing $r(y) + |D_y|$. Such a choice exists in a finite nonempty set. Write H_y for the high offsets at least y.

Lemma 9.1 (Anchor balance)**.** *For high $b \le y \le c$,*

$$\begin{aligned} r(y) - r(b) &\le |D_b \setminus D_y|, \\ |D_y \setminus D_c| &\le r(c) - r(y). \end{aligned}$$

Moreover, for $b \in D_y$ and a high $c \ge y$ with $b + c \ge N$, let $j = |\{v \in D_y : v > b\}|$. The jth element c' of H_y in increasing order (starting at index zero) exists and satisfies $c' < c$ and $b + c' < N$.

Proof. The first two inequalities are the minimizing inequality, together with $D_y \subseteq D_b$ and $D_c \subseteq D_y$ respectively. For the final assertion choose the smallest $r \in H_y$ with $b + r \ge N$; it exists because c is such an element. Every $v \in D_y$ with $v \ge b$ lies in $D_y \setminus D_r$. There are $j + 1$ such elements. The second balance inequality therefore shows that at least $j + 1$ elements of H_y are below r. Thus its jth element c' is below $r \le c$. By minimality of r, $b + c' < N$. □

Formal correspondence: `an_bal_left; an_bal_right`.

Formal correspondence: `an_colrank`.

Set

$$g = N - y, \quad I = [0, g), \quad J = [y, m), \quad P_* = Q_2 \cap I, \quad R_* = Q_3 \cap [0, y),$$

and

$$T_* = y + (Y_3 \cap I), \qquad F = J \setminus (Q_3 \cup T_*).$$

These are integer intervals and finite sets of integer positions.

Lemma 9.2 (Shell count)**.** *With $p = m - |\mathcal{B}_4|$,*

$$|Q_3| + |F| \le |R_*| + |P_*| + \rho + z + |N_{40}| + |Q_0| + |Q_1 \cap I| + p. \tag{18}$$

Proof. First $Q_3 \cap I = \varnothing$. To see this, write the canonical triple of $q \in Q_3$ in sorted order $a \le b \le c$. All factors are at most q. The pair $a + b$ is wrapped, so b is high and $q \ge b \ge N/2$. Since y is high, $g = N - y \le N/2$, whence $q \notin I$.

The sets F, $Q_3 \cap J$, and T_* partition J. Indeed $T_* \subseteq Y_4 \cap J$, and Q is disjoint from Y_4. Translation preserves cardinality, so

$$|Q_3| + |F| = |R_*| + m - y - |Y_3 \cap I| = |R_*| + \rho + |I \setminus Y_3|.$$

By (9), every position outside Y_3 belongs to $\widehat{Z}$, N_{40}, Q, or the p missing positions of $\mathcal{B}_4$. In I, the Q contribution is contained in $Q_0 \cup (Q_1 \cap I) \cup P_*$ because $Q_3 \cap I = \varnothing$. Taking cardinalities gives (18). □

Formal correspondence: `an_shell`.

Lemma 9.3 (Payment count)**.** *For the minimizing anchor y,*

$$|P_*| + |R_*| \le A + |F| + |Q_1 \setminus I|. \tag{19}$$

Proof. We construct two injective maps to pair-direction incidences and then treat their used incidences separately.

Lower cubics. Let $q \in R_*$ have sorted canonical triple $a \le b \le c$. Since $w(q) = 3$, all three factors are at most $q < y$. The parent obtained by removing c is the canonical wrapped pair (a, b) at $z = q - c$. Since $a + b \ge N$, both b and c are high. The first balance inequality shows that

$$r(c) - r(b) < |D_b \setminus D_y|.$$

Choose u to be the element of $D_b \setminus D_y$ at that index, in increasing order. Send q to the incidence (z, u). This map is injective: the position z determines its canonical pair (a, b), then u determines its rank in $D_b \setminus D_y$, hence determines $r(c)$ and c, and finally $q = z + c$. Also $u + b < N$ and b is high, so u is low.

If the incidence is used, its target $q' = z + u$ has canonical triple (u, a, b). Indeed $a + b \ge N$ but $u + b < N$, so $u < a \le b$. The wrapped factor count is one: $q' - u = z \ge 0$, while

$$q' - a = b + u - N < 0, \qquad q' - b = a + u - N < 0.$$

Furthermore $u \notin D_y$ gives $u + y \ge N$, and $q' = z + u \ge u \ge g$, so $q' \in Q_1 \setminus I$. This map on used incidences is injective because the canonical target triple recovers its smallest factor u, hence $z = q' - u$; injectivity of the incidence map then recovers q.

Prefix cubics. Let $q \in P_*$ have sorted canonical triple $a \le b \le c$. The count $w(q) = 2$ means $a, b \le q < c$, with multiplicities respected. Its parent after removing b is the canonical wrapped pair (a, c) at $z = q - b$. In particular $a + c \ge N$. As $b \le q < g$, we have $b \in D_y$. Also $c \ge y$ and c is high: $c \ge N - a \ge N - b > y$. Apply the final assertion of Lemma 9.1 to obtain

$$c' = \text{the } |\{v \in D_y : v > b\}|\text{th high offset at least } y,$$

with $c' < c$ and $b + c' < N$. Send q to (z, c'). This incidence map is injective: c' determines its rank in H_y, hence the number of elements of D_y greater than b, which uniquely determines $b \in D_y$; then $q = z + b$. Its direction c' is high, so its image is disjoint from that of the lower-cubic map.

If this incidence is used, the target $q' = z + c'$ has canonical triple (a, c', c). We have $a \le b < N - y \le y \le c'$, and in fact $a < c' < c$. Since $a + c \ge N$ and $a + c' < N$, the two smaller factors are at most q' and $c > q'$. Thus $q' \in Q_2$. Also $q' \ge c' \ge y$, so $q' \in Q_2 \cap J$. The target map is injective because the middle factor of its sorted canonical triple is c'; this recovers the incidence and hence its source. Finally $Q_2 \cap J \subseteq F$: it is disjoint from Q_3 and from $T_* \subseteq Y_4$.

Every source therefore gives either a distinct unused incidence, a distinct point of $Q_1 \setminus I$, or a distinct point of F. The two unused-incidence images are disjoint by their low/high directions. Every unused incidence has weight at least one, so their total count is at most A. This proves (19). □

Formal correspondence: `an_R_exists; an_P_exists`.

Formal correspondence: `an_R_inj; an_P_inj`.

Formal correspondence: `an_PR_ne; an_R_target`.

Formal correspondence: `an_P_target; an_R_target_inj`.

Formal correspondence: `an_P_target_inj; an_payment`.

Theorem 9.4 (Strip-free anchor bound)**.** *If the window model has a high offset, then*

$$W_0 + p \ge \mathrm{coll} + 2\delta + d_4 - h_4.$$

Proof. Add (18) and (19) to eliminate $|P_*| + |R_*|$ and $|F|$. Use $|Q_1 \cap I| + |Q_1 \setminus I| = |Q_1|$ to obtain

$$|Q_3| \le A + \rho + z + |N_{40}| + |Q_0| + |Q_1| + p.$$

Substitute this into (17). The result is

$$W_0 + p \ge \mathrm{coll} + 2\delta + d_4 - |N_{40}| + |Q_0| + 4\eta.$$

Now $|N_{40}| \le h_4$ and the last two terms are nonnegative. □

Formal correspondence: `an_anchor_bound`.

Corollary 9.5. *Every strip-free window model satisfying (15) has $W_0 + p \ge 0$. If its conductor is $4m - \rho$ and it is regarded as the semigroup T of Section 3, then it satisfies Wilf's inequality.*

Proof. There is a high offset because $h \ge 3$. Combine Theorem 9.4, coll ≥ 0, Lemma 6.1, and Theorem 7.2. For the semigroup conclusion, $\ell \ge 4 + 3K \ge 16$ by (10). Thus

$$W = W_0 + p(\ell - 4) = (W_0 + p) + p(\ell - 5) \ge 0.$$

□

Formal correspondence: `an_anchor_core; wm_anchor_core`.

10. The extension to strip offsets

A full first-level generated core can have offsets in $G = [N, m) \cap B$. Let $K = |O|$ and $\sigma = |G|$. Write Y_t for the window sets, and Z_t^B for the once-wrapped sets formed from all offsets B. Also put

$$V_t = \{s + \rho - 2m : s \in t(\{0\} \cup B),\ 2m \le s + \rho < 3m\}.$$

For $k \le 4$ the level formula is

$$\mathcal{B}_k = Y_k \cup Z_{k-1}^B \cup V_{k-2}, \tag{20}$$

with negative subscripts interpreted as empty. Indeed the quotient j in (5) is at most two: $j \ge 3$ would force $t \le 1$ while $s + \rho < 2m$. Conversely every set on the right supplies the corresponding

representation. Unwrapped sums cannot contain a strip offset, so their definition uses O alone. Also $V_0 = V_1 = \varnothing$.

The single strip offsets give

$$G' = Z_1^B = \{g + \rho - m : g \in G\} \subseteq [0, \rho), \qquad |G'| = \sigma.$$

Define the following disjoint parts of the newly reached positions:

$$\begin{aligned} Z_* &= Z_2^B \setminus (Y_3 \cup G'), \\ Z_W &= \widehat{Z}(O) \setminus G', & Z_S &= Z_* \setminus Z_W, \\ Q_* &= (Z_3^B \cup V_2) \setminus (Y_4 \cup Z_2^B), \\ Q_D &= Q_* \cap V_2, & Q^{(3)} &= Q_* \setminus V_2, \\ Q_W &= Q^{(3)} \cap Z_3(O), & Q_S &= Q^{(3)} \setminus Z_3(O), \\ N_* &= (Y_4 \setminus Y_3) \setminus Z_2^B. \end{aligned}$$

Here $Z_W \subseteq Z_*$: a pure-window wrapped pair outside Y_3 and G' is certainly a full-model pair outside those sets. It follows directly from (20) that

$$\begin{aligned} \ell &= 4 + 3K + 2h_2 + h_3 + 2\sigma + z_W + z_S, \\ d &= K + 1 + h_2 + h_3 + \sigma + z_W + z_S + r + q_W + q_S + q_D, \end{aligned} \tag{21}$$

where lower-case letters are the cardinalities of the corresponding sets and $r = |N_*| \le h_4$. In particular $Q_W \subseteq Q(O)$.

Use canonical representations over O for Q_W, Z_W, and over B for Q_S, Z_S. A value having a pair representation entirely in O has a canonical pair over B entirely in O. To check this, every strip coordinate occurs after every window coordinate in the sorted enumeration. If a strip coordinate has index j, its contribution to the base-6 code is at least 6^j, whereas the code of a pair on smaller indices is at most $2 \cdot 6^{j-1} < 6^j$. Minimality excludes that strip coordinate.

Wrapped parents of a canonical cubic in Q_W lie in Z_W. The proof in Section 5 first puts them in $\widehat{Z}(O)$. Such a parent cannot lie in G', because adding the removed window offset would then represent the cubic position using a strip offset and one window offset, placing it in Z_2^B, contrary to the definition of Q_*. Thus the two anchor injections and all their injectivity proofs apply unchanged to this restricted window pair/cubic system.

Write $Q_j^W = \{q \in Q_W : w(q) = j\}$ and $q_j = |Q_j^W|$. Let U be the *unweighted* number of unused window-direction incidences at Z_W. If

$$s_W = \sum_{q \in Q_W} w(q), \qquad \delta_W = (K+2)h_2 - 3h_3 - 3q_W + s_W,$$

then

$$s_W + U \le (K+2)z_W, \qquad \delta_W \ge \delta(O). \tag{22}$$

The first assertion follows by assigning weight $\kappa_{z,i} + 1$ to used incidences and weight one to unused incidences; their combined weight is no greater than the available $K + 2$ at each pair. The second follows from $Q_W \subseteq Q(O)$ and $3 - w(q) \ge 0$.

Choose the same minimizing anchor y for O. Put

$$F_* = J \setminus (Q_3^W \cup T_*), \qquad F_0 = F_* \setminus Q_2^W, \qquad S_I = (G' \sqcup Z_S \sqcup Q_D \sqcup Q_S) \cap I,$$

and write $s_I = |S_I|$, $f_0 = |F_0|$. The shell proof and payment proof from Section 9, retaining the extra positions in S_I, yield

$$q_3 + f_0 \le U + \rho + z_W + r + q_0 + q_1 + p + s_I. \tag{23}$$

For precision, the shell includes s_I because the complement of Y_3 now has the extra disjoint pieces G', Z_S, Q_D, Q_S. Payment uses only $Q_2^W \cap J \subseteq F_* \cap Q_2^W$, so the unused remainder is exactly F_0. These are the only modifications to the proofs already given.

Formal correspondence: `st_selW; st_W_count`.

Formal correspondence: `st_d3pp_ge; st_shell`.

Formal correspondence: `st_payment`.

10.1. **Strip cubics and their demand slots.** For $q \in Q_S$, the canonical degree-three representation β_q over B contains a strip factor; otherwise $q \in Z_3(O)$. If a factor g is greater than q, its cofactor has value $q+m-\rho-g < N$, so both cofactor offsets lie in O. Therefore g must be a strip factor, since otherwise the full cubic would be pure window. The cofactor position

$$u = q + m - g$$

belongs to $Y_2 \setminus Y_1$: a representation using at most one offset would put q in Z_2^B.

There is at most one factor greater than q, counted with multiplicity. After removing any such factor the remaining factors are window offsets; a second such factor would also have to be a strip offset by the same argument, which is impossible. Partition Q_S as $Q_{h0} \sqcup Q_{\mathrm{top}}$ according as such a factor exists or not. Then

$$4q_S \le \sum_{q \in Q_S} w_B(q) + |Q_{\mathrm{top}}| + 2|Q_{h0}|. \tag{24}$$

In fact equality holds: $w_B(q) = 2$ on the first part and 3 on the second. Moreover every $q \in Q_{h0}$ is in the image of

$$\psi : (Y_2 \setminus Y_1) \times G \longrightarrow \mathbb{Z}, \qquad \psi(u, g) = u + g - m, \tag{25}$$

and hence $|Q_{h0}| \le \sigma h_2$.

At a vertex $z \in Z_S$, provide slots (z, i, c) for every offset index i of B and $0 \le c \le \kappa_{z,i}$. There are $K + \sigma + 2$ slots at each such vertex. At $z \in Z_W$ provide one slot $(z, i, 0)$ for every strip direction i, giving σ slots. Total capacity is

$$C_* = (K + \sigma + 2)z_S + \sigma z_W. \tag{26}$$

The demand consists of triples (q, i, c) with $q \in Q_S$, $a_i \le q$, and $0 \le c < \beta_{q,i}$. Its size is $\sum_{q \in Q_S} w_B(q)$. Map demand to slots by

$$(q, i, c) \longmapsto (q - a_i, i, c).$$

This is injective, since i, c and the target position recover q. Canonical parent closure puts the parent in Z_*; it cannot lie in Y_3 or G' because those memberships would put q in Y_4 or Z_2^B. At a Z_S parent, $\beta_{q,i} = \kappa_{q-a_i,i} + 1$, so the slot exists. At a Z_W parent, purity of its canonical pair forces the added factor to be a strip and its multiplicity to be one: otherwise the cubic would be pure window. Thus the corresponding slot exists there too.

Additionally reserve a slot $(z, i, 0)$ whenever $z \in Z_*$, $a_i \in G$, and

$$z + a_i \ge m, \qquad z + a_i - m \in Q_W.$$

Call the set of these reserved slots A_4. They are available in both vertex types. They are disjoint from the demand image: a demand image has $z + a_i = q < m$, whereas reserved slots require $z + a_i \ge m$. Consequently

$$\sum_{q \in Q_S} w_B(q) + |A_4| \le C_*. \tag{27}$$

Formal correspondence: `st_QS_nonW; st_QS_nw`.

Formal correspondence: `st_QS_four; st_Qh0_sub`.

Formal correspondence: `st_slot_bound`.

10.2. **The good and bad shifted positions.** Call $s \in S_I$ good when $s + y \notin Q_W$, and bad otherwise. Let B_2 be the bad positions for which some $g \in G$ has

$$u = s + y + m - g \in Y_2 \setminus Y_1.$$

Every $s \in S_I$ has a strip offset g for which

$$s + y + m - g \in Y_3 \cup Z_2^B. \tag{28}$$

Here is a verification for all four components of S_I. If $s \in G'$, remove its defining strip offset; the remaining position is $y + \rho \in Y_1 \subseteq Y_3$. If $s \in Z_S$, choose a two-offset representation. At least one factor is a strip, since a pure-window pair would put it in Z_W. Remove that strip g and add y. The remaining pair is unwrapped: $s+y < N \le g$ gives $s+y+m-g < m$. If $s \in Q_D$, its twice-wrapped representation is a pair of strip offsets; removing one and adding y leaves a once-wrapped pair, with position in Z_2^B. Finally, if $s \in Q_S$, choose a strip factor. Since $s < g_0 = N - y < N$ for the interval endpoint g_0, that factor is greater than s. The preceding cubic analysis leaves an unwrapped window pair; adding y is still unwrapped, again because $s + y < N \le g$. This proves (28).

For a bad s, let $x = s + y \in Q_W$ and choose g as in (28), with $u = x + m - g$. Then $u \notin Y_1$ and $u \notin G'$. The first membership would express x as a once-wrapped sum of at most two offsets, contrary to $x \notin Z_2^B$. The second would express x as a twice-wrapped strip pair, contrary to $x \notin V_2$. If $u \in Y_2$, this is a member of B_2. If $u \in Y_3 \setminus Y_2$, the pair (u, g) lies in a pool of size σh_3 and determines $s = u + g - m - y$. Otherwise $u \in Z_*$, and $(u, g, 0)$ is a reserved slot in A_4, also determining s. Therefore

$$|\mathrm{bad}| \le |B_2| + \sigma h_3 + |A_4|. \tag{29}$$

The sets $y + B_2$ and Q_{h0} are disjoint: the former lies in Q_W and the latter in Q_S. Both are images under (25), so

$$|B_2| + |Q_{h0}| \le \sigma h_2. \tag{30}$$

Partition $Z_S = Z_{CC} \sqcup Z_{BC}$, where Z_{CC} consists of positions having a representation by two strip offsets. The remaining positions have a mixed window-strip pair representation, so

$$|Z_{BC}| \le K\sigma. \tag{31}$$

Also

$$|Z_{CC}| + q_D \le \frac{\sigma(\sigma+1)}{2}. \tag{32}$$

Indeed the disjoint sets $Z_{CC} \subseteq Z_2^B$ and $Q_D \cap Z_2^B = \varnothing$ are both contained in the image of unordered strip pairs under $(a, b) \mapsto (a + b + \rho) \bmod m$. A twice-wrapped pair necessarily consists of two strips, since either offset below N would make $a + b + \rho < 2m$.

Finally the three sets $y + \mathrm{good}$, Z_{CC}, and Q_{top} are disjoint subsets of F_0. For the first set, $s \notin Y_3$ rules out $s + y \in T_*$, while goodness excludes Q_W. It lies below N because $s \in I$. Both other sets lie at or above N: for Z_{CC} this follows from $a, b \ge N$, and for Q_{top} from the presence of a strip factor no greater than the position. They avoid $T_* \subseteq [0, N)$ and Q_W. They are disjoint from each other because $Z_{CC} \subseteq Z_2^B$ but $Q_{\mathrm{top}} \subseteq Q_*$ avoids Z_2^B. All three sets lie in J, since $y < N$. Therefore

$$|\mathrm{good}| + |Z_{CC}| + |Q_{\mathrm{top}}| \le f_0. \tag{33}$$

Formal correspondence: `st_SI_exists; st_bad_u`.

Formal correspondence: `st_bad_bound; st_bad2_bound`.

Formal correspondence: `st_ZBC_bound; st_ZCC_QD`.

Formal correspondence: `st_good_bound`.

Lemma 10.1 (Strip estimate)**.** *The extra positions satisfy*

$$s_I + 5z_S + 4q_D + 4q_S \le \sigma h_3 + 2\sigma h_2 + 5K\sigma + 2\sigma(\sigma+1) + C_* + f_0. \tag{34}$$

Proof. By (24) and (27),

$$4q_S \le C_* - |A_4| + |Q_{\text{top}}| + 2|Q_{h0}|.$$

Use $s_I = |\text{good}| + |\text{bad}|$, then (29) and (33), to deduce

$$s_I + 4q_S \le f_0 - |Z_{CC}| + \sigma h_3 + C_* + |B_2| + 2|Q_{h0}|.$$

By (30) and $|Q_{h0}| \le \sigma h_2$, the final two terms are at most $2\sigma h_2$. Adding $5z_S + 4q_D$ gives an additional

$$5z_S - |Z_{CC}| + 4q_D = 5|Z_{BC}| + 4(|Z_{CC}| + q_D),$$

which is bounded by $5K\sigma + 2\sigma(\sigma+1)$ using (31)–(32). □

Formal correspondence: `st_strip_lemma`.

Theorem 10.2 (Full-model anchor bound)**.** *Whenever O contains a high offset,*

$$W_0(B) + p(B) \ge \text{coll}(O) + 2\delta(O) + d_4(O) - h_4(O). \tag{35}$$

Proof. We provide the cancellation explicitly. Put $z_0 = |\widehat{Z}(O)|$, so $z_W \le z_0$. Replace $\delta(O)$ on the right by the larger δ_W from (22), and call the resulting right side C. Expand (21) and the definitions of coll, δ_W, d_4. The difference is

$$\begin{aligned} W_0 + p - C = {} & 6z_0 + (K-3)(z_W + z_S) - 4r + 2q_W - 2s_W \\ & - 4q_S - 4q_D + \rho + p + 5h_4 \\ & + \sigma(5K + 2 + 2\sigma + 2h_2 + h_3 + z_W + z_S). \end{aligned}$$

Since $s_W = q_1 + 2q_2 + 3q_3$ and $q_W = q_0 + q_1 + q_2 + q_3$, the first inequality of (22) yields

$$2q_W - 2s_W \ge 2q_0 + q_1 + U - (K+2)z_W - q_3.$$

Now use (23). The difference is at least

$$\begin{aligned} & 6(z_0 - z_W) + (K-3)z_S - 5r + q_0 - s_I + f_0 - 4q_S - 4q_D + 5h_4 \\ & \quad + \sigma(5K + 2 + 2\sigma + 2h_2 + h_3 + z_W + z_S). \end{aligned}$$

Apply (34), substitute $C_* = (K + \sigma + 2)z_S + \sigma z_W$, and cancel. The lower bound becomes exactly

$$6(z_0 - z_W) + 5(h_4 - r) + q_0 \ge 0.$$

This proves the result, since $\delta_W \ge \delta(O)$. □

Formal correspondence: `st_A29A`.

Corollary 10.3. *Every full-offset model whose window offsets satisfy (15) has*

$$W_0(B) + p(B) \ge 0, \tag{36}$$

and consequently

$$13W_0(B) + 23p(B) \ge 0. \tag{37}$$

Proof. Use Theorem 10.2, the collision and cubic nonnegativity, and Theorem 7.2. Then (37) follows from $13W_0 + 23p = 13(W_0 + p) + 10p$ and $p \ge 0$. □

Formal correspondence: `st_core; wm_model_core`.

Formal correspondence: `cr_CORE_proof`.

11. Graded Apéry counts

We now give the counting infrastructure for arbitrary depth-four semigroups. This section uses new local letters; in particular h here counts strip primitives, not high window offsets. Let

$$\mathrm{Ap}(S,m)=\{x\in S : x-m\notin S\},\qquad X_i=\mathrm{Ap}(S,m)\cap[im-\rho,(i+1)m-\rho).$$

Interpret the subtraction test in $\mathbb{Z}$. There is exactly one Apéry element in each residue class modulo m: choose the least semigroup element in that class. Every Apéry element is below $c+m$. Its translates by nonnegative multiples of m are precisely the elements of that residue class in S. Consequently

$$m=\sum_{i=0}^{4}|X_i|,\qquad \ell=4+3|X_1|+2|X_2|+|X_3|,\qquad X_0=\{0\}. \tag{38}$$

Every primitive other than m is Apéry. Put $K=|X_1|$. These K elements are primitive, lying in $[m,2m-\rho)$. Split level-two primitives into h strip elements in $[2m-\rho,2m)$ and k ordinary elements in $[2m,3m-\rho)$. Let z count level-three primitives.

Represent each decomposable Apéry element as a sum of left primitives other than m, first minimizing the number of factors and then the base-six code of its exponent vector. Such a representation exists by primitive generation; no factor m can occur in an Apéry element. It has at most four factors, since the element is below $c+m\le 5m$. Every parent obtained by deleting a factor is again Apéry: a predecessor by m would give one for the original element. It is canonical, since either a shorter representation or a smaller code for the parent would improve the original. Thus all the weighted shadow counts of Lemma 5.3 apply to these canonical monomials.

Use letters B,C,D for factors of levels one, two, three, respectively. The following table defines counts of decomposable Apéry elements by their canonical factor type; the listed types exhaust the possibilities.

level	factor types	counts
2	BB	b
3	BB, BBB, BC, CC	w, v, x, f
4	$BBB, BBBB, BC, CC, BBC, BCC, CCC, BD, CD$	$t, u, y, d, H_0+H_1, j, g, r, s$

Indeed factors of types B,C,D are at least $m,2m-\rho,3m-\rho$, respectively, and adding any positive factor raises the level by at least one. Pure BB sums are below c; five factors exceed the Apéry upper bound; and four-factor types containing C already exceed that bound. These observations give the table by enumerating degrees two through four. In BBC, put the value in H_0 if its BB sub-sum is below $3m-\rho$, otherwise in H_1. Let w_ℓ count the w values below $3m$. Split $x=x_h+x_o$, putting in x_h every x value with *some* representation as a first-level primitive plus a strip primitive, and putting the remaining values in x_o. This definition does not require that representation to be canonical.

Write R_4 for the total number of decomposable Apéry elements of level four. With $n=1+K+h+k+z$, the counts give

$$\ell=4+3K+2(h+k)+z+2b+w+v+x_h+x_o+f, \tag{39}$$

$$d=n+b+w+v+x_h+x_o+f+R_4, \tag{40}$$

$$R_4=t+u+y+d_{CC}+H_0+H_1+j+g+r+s.$$

Hereafter d_{CC} denotes the table's CC count, to avoid confusing it with the conductor-interval count d.

Lemma 11.1. *All the following expressions are nonnegative:*

P	$K(K+1)-2(b+w)$
T	$(K+2)(b+w)-3(v+t)$
V	$(K+2)b-3v$
U	$(K+3)v-4u$
B_h	$Kh-x_h$
B_o	$Kk-x_o-y$
C	$(h+k)(h+k+1)-2(f+d_{CC})$
D	$Kz-r$
E	$(h+k)z-s$
G	$(h+k)b-H_0$
M	$(h+k)(b+w_\ell)+(K+h+k+2)(x_h+x_o+f)-3(H_0+H_1+j+g)$
M'	$kb+h(b+w_\ell)+(K+h+k+2)(x_h+x_o+f)-3(H_0+H_1+j+g)$
S	$\rho-f-H_1-j-g-s.$

Also $0 \le w_\ell \le w$.

Proof. Unordered pair counts prove $\mathsf{P},\mathsf{C} \ge 0$, and ordered pair counts prove $\mathsf{D},\mathsf{E} \ge 0$. Weighted canonical-parent counting proves $\mathsf{T},\mathsf{V},\mathsf{U} \ge 0$: each cubic supplies three parent incidences, each pair has total capacity $K+2$, and each triple has capacity $K+3$ for quartics. The level bounds put the parents of a left cubic in b and those of a quartic in v.

The definition of x_h gives $x_h \le Kh$. Every remaining BC value counted by x_o or y has a representation using an ordinary level-two primitive. For y this follows because a first-level primitive plus a strip primitive is below $4m-\rho = c$. Since x_o and y occupy distinct levels, their sum is bounded by the Kk possible pairs. Deleting the unique C factor of an H_0 monomial gives a pair counted by b, proving $\mathsf{G} \ge 0$.

For the mixed shadows, consider all canonical cubics with at least one C factor. Deleting a factor gives either a pure BB parent in b or w_ℓ, or a parent of type BC or CC counted by x_h+x_o+f. A pure parent can occur only on deleting a C factor. Its value is below $3m$, since the original Apéry value is below $5m-\rho$ and the removed factor is at least $2m-\rho$. Thus its capacity is $h+k$; a mixed pair's total capacity is $K+h+k+2$. Summing gives $\mathsf{M} \ge 0$. For the refinement, a pure parent counted by w_ℓ cannot be extended by an ordinary factor: its value is at least $3m-\rho$, so the sum would be at least $5m-\rho$. Its capacity is only h, yielding $\mathsf{M}' \ge 0$.

Finally the H_1, j, g, s values all lie in $[c+m-\rho, c+m)$, by their factor lower bounds and the definition of H_1. The f values shifted by m lie in the same interval: a CC sum is at least $4m-2\rho$, and a left value is below c. These shifted values are not Apéry whereas the other four classes are. All five classes are disjoint, so their total size is at most the interval length ρ. This proves $\mathsf{S} \ge 0$. □

Formal correspondence: `w23_gm_type_table; w23_gm_general_counts`.

Formal correspondence: `w23_gm_mixed_card; w23_gm_strip_card; cr_gm_mixed_card_split`.

11.1. **Consequences of a negative core deficit.** Suppose $W_0(S) < 0$. Put $a_0 = K+1$ and $P' = 2b+w+v+x_h-3h$. Then

$$3(k+z)+3 \le a_0, \tag{41}$$

$$9a_0(a_0-1)+(4a_0+16)b \le (a_0+16)P', \tag{42}$$

$$6(a_0+2)b - a_0(a_0-1)(a_0-8) \le 12P' \tag{43}$$

where the last two assertions require $k+z\ge 1$. Here are explicit algebraic certificates. They can be verified by expanding (39)–(40); no optimization assumption is needed. Define

$$p_1=kv,\quad p_2=h(w-w_\ell+v),\quad p_3=z(1+z+2b+v+x_h+x_o+f),$$
$$p_4=(3(k+z)-K+1)w,\quad p_5=(h+k)(w-w_\ell+v),$$
$$p_6=z(1+z+2b+w+v+x_h+x_o+f),$$
$$p_7=(12a_0-63)h,\quad p_8=(K+2)x_h,\quad p_9=(K+11)v,$$
$$p_{10}=(12a_0-48)h,\quad p_{11}=(K+11)\mathsf{P}.$$

If (41) fails, integrality gives $3(k+z)-K+1\ge 0$, so every term in

$$W_0=3\mathsf{P}+\tfrac{4}{3}\mathsf{T}+\tfrac{2}{3}\mathsf{V}+\mathsf{U}+5\mathsf{B}_h+5\mathsf{B}_o+2\mathsf{C}+4\mathsf{D}+3\mathsf{E}$$
$$+\mathsf{G}+\mathsf{M}'+\mathsf{S}+y+p_1+p_2+p_3+\tfrac{1}{3}p_4$$

is nonnegative, a contradiction. This proves (41). When $k+z\ge 1$, it implies $a_0\ge 6$. The slacks in (42) and (43), respectively, equal

$$4\mathsf{T}+3\mathsf{U}+15\mathsf{B}_o+6\mathsf{C}+12\mathsf{D}+9\mathsf{E}+3\mathsf{G}+3\mathsf{M}+3\mathsf{S}+3y$$
$$+3p_5+3p_6+p_7+p_8+p_9-3W_0,$$
$$8\mathsf{T}+6\mathsf{U}+18\mathsf{B}_h+30\mathsf{B}_o+12\mathsf{C}+24\mathsf{D}+18\mathsf{E}+6\mathsf{G}+6\mathsf{M}+6\mathsf{S}$$
$$+6y+6p_5+6p_6+p_{10}+p_{11}-6W_0.$$

Every term is nonnegative. This proves the asserted numerical bounds.

Formal correspondence: `cr_S_arith; cr_S_bounds`.

12. From generated cores to arbitrary semigroups

For a depth-four numerical semigroup S, let

$$A=P(S)\cap[m,2m),\qquad T=\langle A\rangle\cup[c,\infty).$$

Then $T\subseteq S$, the multiplicities are equal, and T has conductor exactly c. The last assertion follows because $c-1\notin S$ and hence $c-1\notin T$, whereas every integer at least c was adjoined. The set A consists of all semigroup elements in $[m,2m)$, because such elements cannot decompose. Consequently T is exactly a full-offset model of Section 3, with $B=\{a-m:a\in A\setminus\{m\}\}$.

For $U=S,T$, use n_U,ℓ_U,d_U,p_U and $W_0(U)$ as in (2)–(3), with $q=4$. Every decomposable conductor-interval element of T remains decomposable in S. Thus $d_T\le d_S$ and

$$p_T-p_S=d_S-d_T. \tag{44}$$

When $W_0(S)<0$ and S has a left primitive outside $[m,2m)$, the source proves the quantitative transfer bound

$$23(d_S-d_T)\le 13(W_0(S)-W_0(T)). \tag{45}$$

This is `cr_thmA`, not a generic monotonicity statement. Inclusion $T\subseteq S$ alone does not imply it. The source establishes it through a classification of canonical monomials, a fibre count for the additional primitives, and an arithmetic inequality at coefficient $23/13$.

The consequence of the transfer estimate is immediate. If $13W_0(T)+23p_T\ge 0$, then (44)–(45) give

$$13W_0(S)+23p_S=13W_0(T)+23p_T$$
$$+13(W_0(S)-W_0(T))-23(d_S-d_T)\ge 0.$$

If $\ell_S\ge 6$, this is enough for Wilf, since

$$13W(S)=13W_0(S)+13p_S(\ell_S-4)\ge 13W_0(S)+26p_S\ge 3p_S\ge 0.$$

If $W_0(S) \geq 0$, identity (4) already proves the result. If every left primitive of S is in $[m, 2m)$, primitive generation shows that $S = T$, so no transfer estimate is needed.

Formal correspondence: `cr_wilf_of_F; cr_core_reduction`.

12.1. **The fibre bridge.** We first prove the combinatorial comparison between S and T. Write

$$C = P(S) \cap [2m, 3m - \rho), \qquad E = P(S) \cap [3m - \rho, 4m - \rho),$$

and let $a = |A|$, $k = |C|$, $z = |E|$. Then $n_T = a$ and $n_S = a + k + z$. For each $c_0 \in C$, define its fibre

$$F_{c_0} = \{x \in A : c_0 + x < c,\ c_0 + x \notin T,\ c_0 \text{ is the largest } c_1 \in C \text{ with } c_0 + x \in c_1 + A\}.$$

Let $r_{c_0} = |F_{c_0}|$, $s_r = \sum_{c_0 \in C} r_{c_0}$, and

$$R = |\{c_0 \in C : c_0 + m \in T\}|, \quad b_0 = |T \cap [2m, 3m - \rho)|, \quad s_{\min} = \sum_{c_0 \in C} \min\{2b_0, r_{c_0}(r_{c_0} + 1)\}.$$

Let $a_0 = |P(S) \cap [m, 2m - \rho)|$. Every fibre lies in this first-level primitive set: $c_0 \geq 2m$ and $c_0 + x < 4m - \rho$ imply $x < 2m - \rho$. Thus $r_{c_0} \leq a_0$.

Lemma 12.1. *These quantities satisfy*

$$\ell_S \geq \ell_T + k + s_r + z, \tag{46}$$

$$2(d_S - d_T + s_r + R) \leq 2ak + k(k + 1) + s_{\min} + 2az. \tag{47}$$

The number of empty fibres is at most R.

Proof. A positive element of S below c that is outside T must have a primitive factor from $C \cup E$. An E factor cannot be accompanied by another positive factor, since that would be at least c. A C factor can be accompanied by at most one factor in A, since three factors including one at least $2m$ have sum at least $4m \geq c$. Two C factors also have sum at least $4m$. Therefore new left elements are in C, E, or $C + A$.

The sets C, E are disjoint and their elements are primitive, whereas $c_0 + x$ is decomposable. Distinct fibre entries have distinct values: equality of two values forces equality of their largest C representative, and then equality of x. All fibre values are outside T by definition. This proves (46).

If $F_{c_0} = \varnothing$, then $c_0 + m \in T$. Otherwise m would belong to its fibre: $c_0 + m < c$, and no $c_1 > c_0$ can represent $c_0 + m = c_1 + x$ with $x \geq m$. Thus each empty fibre contributes to R.

Consider a new decomposable $y \in [c, c + m)$ of S which is not decomposable in T. Such a y is primitive in T. A positive decomposition $y = u + v$ must have at least one summand outside T, so the preceding classification and the bounds $y < 5m - \rho$, $C \subseteq [2m, 3m - \rho)$, and $E \subseteq [3m - \rho, 4m - \rho)$ imply that y has one of the forms

$$E + A, \qquad C + A, \qquad C + C, \qquad C + A + A.$$

For example, if a new left summand lies in E, the other is below $2m$ and hence lies in A. If it lies in C and the other summand is not primitive, that summand splits into two elements below $2m$, both in A. If a summand lies in $C + A$, the remaining summand is below $2m$ for the same size reason.

The $E + A$ values number at most az. There are at most $k(k+1)/2$ values in $C + C$. For $C + A$, there are ak pairs in total; at least $s_r + R$ of these pairs have sums below c. The s_r fibre pairs have left sums outside T, while the R pairs (c_0, m) have left sums in T, so those pairs are distinct. Thus at most $ak - s_r - R$ pairs can contribute to the conductor interval.

Finally consider a new right value with a $C + A + A$ representation. Assign it to the largest $c_0 \in C$ admitting such a representation, and write $y = c_0 + x + x'$. Both x, x' lie in F_{c_0}. Indeed $c_0 + x < c$ since $y < c + m$ and $x' \geq m$. If $c_0 + x \in T$, then $y = (c_0 + x) + x'$ decomposes in T,

impossible. A representation $c_0 + x = c_1 + u$ with $c_1 > c_0$ and $u \in A$ contradicts maximality of c_0 in the triple representation. The same reasoning applies to x'.

For each assigned c_0, there are at most $r_{c_0}(r_{c_0}+1)/2$ unordered fibre pairs. There are also at most b_0 possible sums $x + x'$, because these sums lie in $T \cap [2m, 3m-\rho)$: the upper bound follows from $c_0 \geq 2m$ and $y < 5m - \rho$. Hence the number of such values assigned to c_0 is bounded by the minimum of those two counts. Summing all four upper bounds, allowing overlap between the categories, gives (47). □

Formal correspondence: `cr_fib_sub_P1; cr_zero_fibres`.

Formal correspondence: `cr_leftCount_bridge; cr_newright_cases`.

Formal correspondence: `cr_caa_fib; cr_right_bridge`.

12.2. **Assembling the quantitative certificates.** We shall use the graded counts of Section 11 to establish a lower bound

$$\ell_T \geq 1 + 3a + P \tag{48}$$

and the two numerical inequalities

$$0 \leq k(26P - 46a - 49(k+1)) + (26(a+k) + 150)s_r - 75s_{\min} + 150R, \tag{49}$$

$$0 \leq z\big(13(1 + P + 2k + z) - 23a\big). \tag{50}$$

Together with Lemma 12.1, these suffice for (45). To verify this without any hidden semigroup argument, set $\Delta = d_S - d_T$. After multiplying (47) by $75/2$, substitute its upper bound for 75Δ into

$$13\big((a+k+z)\ell_S - a\ell_T\big) - 75\Delta.$$

Use (46) and (48). Twice the resulting lower bound is the sum of the right sides of (49) and twice (50), together with the extra nonnegative term $26zs_r$. It is therefore nonnegative. This is exactly (45), since the common ρ cancels in $W_0(S) - W_0(T)$.

12.3. **The scalar fibre inequality.**

Lemma 12.2. *Let a, k, β, P, r be integers with $a \geq 6$, $k \geq 1$, $3k + 3 \leq a$, $\beta \geq 0$, and $0 \leq r \leq a$. Suppose*

$$9a(a-1) + (4a+16)\beta \leq (a+16)P, \qquad 6(a+2)\beta - a(a-1)(a-8) \leq 12P.$$

For $M = \min\{2a + 2\beta, r(r+1)\}$ one has

$$26P - 46a - 49(k+1) + (26a + 26k + 150)r - 75M + 150\mathbf{1}_{r=0} \geq 0. \tag{51}$$

Proof. Call the left side E. The proof is an elementary piecewise polynomial estimate; details of all ranges follow.

For $r = 0$, use $M = 0$, the first bound for P, $\beta \geq 0$, and $k + 1 \leq a/3$. They give

$$3(a+16)E \geq 515a^2 - 3244a + 7200 > 0.$$

The quadratic is positive for all real a, as its leading coefficient is positive and its discriminant is negative.

For $r = 1, 2, 3$, use $M \leq r(r+1)$ and the first bound for P after discarding β. The coefficient of k in E is $26r - 49$; use $k \leq (a-3)/3$ for $r = 1$ and $k \geq 1$ for $r = 2, 3$. On writing $v = a - 6 \geq 0$, the resulting lower bounds for $3(a+16)E$ are respectively

$$619v^2 + 5320v + 8388, \quad 720v^2 + 7638v + 10500, \quad 798v^2 + 9000v + 2712.$$

For $6 \leq a \leq 10$ and $4 \leq r \leq a$, the finitely many integer pairs (k, r) can be checked without an unbounded search in β. Set

$$P_1(\beta) = \frac{9a(a-1) + (4a+16)\beta}{a+16}, \quad P_2(\beta) = \frac{6(a+2)\beta - a(a-1)(a-8)}{12}.$$

Replace P by $\max(P_1, P_2)$ and M by its defining minimum. The resulting function of real $\beta \geq 0$ is piecewise linear. Its breakpoints are 0, the nonnegative solution of $P_1 = P_2$, and the nonnegative number $(r(r+1) - 2a)/2$. Beyond the last breakpoint it is increasing. Substitution at those at most three points, for $1 \leq k \leq \lfloor (a-3)/3 \rfloor$ and $4 \leq r \leq a$, gives these exact minima over all k, r, β:

a	6	7	8	9	10
$\min E$	0	2036/23	114	1432/5	334

This table is a finite rational check of the displayed formulas; it allows real P, β and therefore also proves the required integer statement.

It remains that $a \geq 11$, $r \geq 4$. First suppose $24r \leq 14a + 21$. The first bound for P, together with the two bounds on M weighted by $52(a+4)$ and $23a + 992$, gives

$$(a+16)E \geq F(a, r),$$

where

$$F = 84a^2 - 1484a - 1568 + (26a^2 + 569a + 1824)r - (23a + 992)r^2.$$

The omitted k contribution is $(a+16)(k-1)(26r-49) \geq 0$. To see $F \geq 0$, put $D = 14a+21-72 > 0$. The identity

$$\begin{aligned} 576DF = {} & 576(14a + 21 - 24r)(162a^2 + 16a - 5024) \\ & + 24(r-3)(4228a^3 + 44716a^2 - 548583a - 421344) \\ & + 24(23a + 992)D(r-3)(14a + 21 - 24r) \end{aligned}$$

has only nonnegative terms: both univariate polynomials have nonnegative coefficients after substituting $a = 11 + v$, $v \geq 0$.

Finally suppose $24r \geq 14a + 22$, the exhaustive complementary integer case. Since $k \geq 1$ and $M \leq 2a + 2\beta$,

$$E \geq 26P - 150\beta - 196a - 98 + (26a + 176)r.$$

If $6\beta \leq (a-1)(a-2)$, the first bound for P gives

$$24(a+16)E \geq 180a^3 + 2388a^2 - 7344a + 8448 > 0.$$

If $6\beta \geq (a-1)(a-2)$, the second bound gives

$$24E \geq 180a^2 - 492a + 528 > 0.$$

For these substitutions, the coefficient of β in the first lower bound is negative, and that in the second is $13a - 124 > 0$; this explains the directions of the split. The displayed polynomials are positive at $a = 11 + v$ coefficient by coefficient. This completes every case. □

Formal correspondence: `cr_CL_r0; cr_CL_r123; cr_CL_small; cr_CL_C1; cr_CL_C2; cr_CL`.

Apply the lemma with $a = a_0$, $\beta = b$, and $P = P' = 2b + w + v + x_h - 3h$, using (41)–(43). For the core T, every Apéry element of S already generated by A remains Apéry in T: a predecessor in T would be one in S. Thus its level-two Apéry set contains the h strip primitives and all b pure pairs; its level-three set contains the disjoint classes w, v, x_h. Equation (38) gives

$$\ell_T \geq 4 + 3K + 2(h+b) + w + v + x_h = 1 + 3a + P, \quad a = a_0 + h, \quad P = P' + 2h.$$

Also $b_0 \leq a_0 + b$. Indeed a core element in $[2m, 3m - \rho)$ either has a predecessor by m in S, which lies among the a_0 first-level primitives including m, or is Apéry in S and a pure pair counted by b. The predecessor lies in $[m, 2m - \rho)$ and hence also in $A \subseteq T$. In the second alternative the element is Apéry in S and decomposable there, and the degree-two type table supplies the claim.

The scalar bound with $M = \min(2a_0 + 2b, r(r+1))$ implies the same bound with $\min(2b_0, r(r+1))$. Replacing a_0 by $a_0 + h$ and P' by $P' + 2h$ adds precisely $h(6 + 26r) \geq 0$. Sum this shifted bound over the k fibres. Lemma 12.1 pays the 150 for each empty fibre from $150R$, giving (49).

For the second certificate, the first bound in (42) and $b \geq 0$ give

$$(a_0 + 16)(13 + 13P' - 23a_0) \geq 94a_0^2 - 472a_0 + 208 > 0 \qquad (a_0 \geq 6).$$

The change to $P = P' + 2h$ and $a = a_0 + h$ adds $3h$, and the extra term $13(2k + z)$ is nonnegative. Multiplication by z proves (50). If $k = 0$, the fibre certificate is empty and the second certificate alone suffices; if $z = 0$, only the fibre argument is needed. The hypothesis of transfer is $k + z \geq 1$, so $a_0 \geq 6$ always holds. This completes the proof of (45).

Formal correspondence: `cr_core_leftCount_ge; cr_b0_le; cr_fibre_arith; cr_stage2_arith; cr_thmA`.

13. The complementary population case

Return to the K first-level Apéry primitives. Call one low if $2a + \rho < 3m$ and high otherwise; let their counts be l, h, so $K = l + h$. These agree with the low and high window offsets after writing $a = m + b$. In this section h again means the high count, not the strip count.

Partition the pure counts of Section 11 by number of high factors:

$$b = b_0 + b_1, \quad w = w_1 + w_2, \quad v = v_0 + v_1, \quad t = t_0 + t_1 + t_2 + t_3, \quad u = u_0 + u_1.$$

The subscript is the number of high factors. A pair of low factors is below $3m - \rho$, so there is no w_0. Two high factors sum to at least $3m - \rho$; adding another positive factor reaches c. Therefore left cubics have at most one high factor; adding two further factors reaches $c + m$, so Apéry quartics also have at most one. This proves the displayed partitions.

Directional weighted shadows give the following nonnegative slacks:

A_0	$l(l+1) - 2b_0$	A_1	$lh - b_1 - w_1$
A_2	$h(h+1) - 2w_2$	T_0	$(l+2)b_0 - 3(v_0 + t_0)$
T_{1a}	$hb_0 - v_1 - t_1$	T_{1b}	$(l+1)(b_1 + w_1) - 2(v_1 + t_1)$
T_{2a}	$lw_2 - t_2$	T_{2b}	$(h+1)(b_1 + w_1) - 2t_2$
T_3	$(h+2)w_2 - 3t_3$	U_0	$(l+3)v_0 - 4u_0$
U_{1a}	$(l+2)v_1 - 3u_1$	U_{1b}	$hv_0 - u_1$.

For example, deleting the unique high factor of a one-high cubic gives an all-low pair, with capacity h, hence $T_{1a} \geq 0$. Deleting its two low factors gives two incidences in mixed pairs, each with low-direction capacity $l + 1$, hence $T_{1b} \geq 0$. The other inequalities follow identically by deleting low or high factors; the three A_i are unordered pair capacities. These arguments also account for repeated factors because the incidence weights are their exponents.

Define the pure surplus

$$F = 3K(K+1) + (2K-2)(b_0 + b_1) + (K-3)(w_1 + w_2 + v_0 + v_1) - 4(t_0 + t_1 + t_2 + t_3 + u_0 + u_1).$$

Lemma 13.1. *If $K(K-1) \leq 3l(l+1)$, then $F \geq 0$.*

Proof. We give the algebraic details, including the small high-count cases. If $h = 0$, all counts with positive subscript vanish; $U_0, T_0, A_0 \geq 0$ give successively

$$F \geq 3l(l+1) + (2l-2)b_0 - 6v_0 - 4t_0 \geq 3l(l+1) - 6b_0 \geq 0.$$

If $h = 1, l = 0$, all counts vanish except w_2, t_3; these are at most $1, w_2$ respectively and $F = 6 - 2w_2 - 4t_3 \geq 0$. If $h = l = 1$, direct expansion gives

$$F = 4A_0 + A_1 + \tfrac{9}{2}A_2 + 2T_0 + 4T_{1a} + 4T_{2a} + \tfrac{4}{3}T_3 + U_0 + U_{1a} + U_{1b} + 3b_1 + 2t_0 \geq 0.$$

For $h = 1, l \geq 2$, the identity

$$\begin{aligned}(l+1)F = {} & 3(l+2)A_0 + \tfrac{3}{2}(l+2)(l+1)A_2 + 2(l+1)T_0 + 2(l+4)T_{1a} + (l-2)T_{1b} \\ & + 4(l+1)T_{2a} + \tfrac{4}{3}(l+1)T_3 + (l+1)(U_0 + U_{1a} + U_{1b}) \\ & + (l+1)(l+2)b_1 + 2(l+1)t_0\end{aligned}$$

proves the claim.

If $h = 2$, the population hypothesis forces $l \geq 1$. Put

$$V = 2(l+1)(b_0 + b_1 + w_1) - 4(t_0 + t_1 + u_0 + u_1) + (l-1)(v_0 + v_1).$$

Expanding gives

$$(l+2)V = (l-1)U_{1b} + (l+3)U_{1a} + (l+2)U_0 + 2(l-1)t_0 + 2(l+1)T_0 + 2(l+2)T_{1b} \geq 0.$$

Then the identity

$$F - (l^2-1) = V + (l+3)(2l - w_1) + (3l + 19/3)(3 - w_2) + 4T_{2a} + \tfrac{4}{3}T_3$$

proves $F \geq 0$, since $w_1 \leq 2l$, $w_2 \leq 3$.

Assume now $h \geq 3$. The population hypothesis forces $l > 0$. Normalize

$$r = \frac{b_0}{l(l+1)/2}, \quad s = \frac{b_1 + w_1}{lh}, \quad t = \frac{w_2}{h(h+1)/2}; \qquad 0 \leq r, s, t \leq 1.$$

The two paired bounds on each of $v_1 + t_1$ and t_2 give

$$v_1 + t_1 \leq \frac{hl(l+1)}{2}\min(r,s), \qquad t_2 \leq \frac{lh(h+1)}{2}\min(s,t).$$

We use the elementary cube fact that

$$\Phi = C + Ar + Bs + Dt - J\min(r,s) - Z\min(s,t)$$

is bounded below by the least of its eight vertex values on $[0,1]^3$. Indeed divide the cube according to the six orders of r, s, t. On each resulting simplex Φ is affine, and its four vertices are cube vertices. For instance, if $r \leq s \leq t$, its value is the convex combination of the values at $000, 001, 011, 111$ with weights $1-t, t-s, s-r, r$.

Set

$$C = 3K(K+1), \quad B = (K-3)lh, \quad D = (K - 3 - 4(h+2)/3)h(h+1)/2,$$
$$J = 2hl(l+1), \quad Z = 2lh(h+1), \quad F_* = (K+1)(3l(l+1) - K(K-1))/6 \geq 0.$$

First suppose $2(l+2) \leq 3h(h-1)$. Put $A = (2K - 2 - 4(l+2)/3)l(l+1)/2$. The quartic bound $4(u_0+u_1) \leq (K+1)(v_0+v_1)$ follows from the identity

$$h((K+1)(v_0+v_1) - 4(u_0+u_1)) = (h-2)U_{1b} + (h+2/3)U_{1a} + hU_0 + \tfrac{1}{3}(3h(h-1) - 2(l+2))v_1.$$

Together with $T_0, T_3 \geq 0$ and the normalized bounds, it yields $F \geq \Phi + (K+1)b_1 \geq \Phi$. All eight vertex values of Φ are nonnegative. The values $C, C+A, C+B$ are immediate. For the others the following identities suffice:

$$C + A + B - J = (3h^2 l + 9h^2 + 6hl + 9h + l^3 + 3l^2 + 2l)/3,$$
$$C + A + D - F_* = hl(h + l + 7), \qquad C + A + B + D - J - Z = F_*.$$

If $l \leq h$, then

$$C + D - F_* = l(3h^2 + 18h - l^2 + 6l + 7)/3 \geq 0,$$
$$C + B + D - Z - F_* = l(l+1)(3h - l + 7)/3 \geq 0.$$

If $l \geq h$, put $y = l - h \geq 0$; then

$$6(C + D) = 2h^3 + 57h^2 + 19h + 18y^2 + (3h^2 + 75h + 18)y,$$
$$6(C + B + D - Z) = 2h^3 + 27h^2 + 19h + (6h + 18)y^2 + (9h^2 + 45h + 18)y.$$

This finishes the first region.

In the other region, $3h(h-1) \le 2(l+2)$ implies $l \ge h$ and $l - 3h + 5 \ge 0$. Put $A = (h-2)(h+3)l(l+1)/2$, keeping the other constants. The improved quartic estimate is

$$4(t_0 + u_0 + u_1) - (K-3)(v_0 + v_1) \le (2(l+2) - h(h-1))b_0 + 4v_1.$$

To verify it, multiply its slack by $l+2$; the result is exactly

$$\begin{aligned}(l - 3h + 5)U_{1b} + (l + h + 1)U_{1a} + (l+2)U_0 \\ + (2(l+2) - 3h(h-1))t_0 + (2(l+2) - h(h-1))T_0,\end{aligned}$$

whose terms are nonnegative. As before this gives $F \ge \Phi + (K+1)b_1$. The previous y identities prove nonnegativity of vertices $001, 011$; $000, 100, 010, 101$ are immediate because $C, A, B \ge 0$. The two remaining vertices satisfy

$$\begin{aligned}2(C + A + B - J) &= h((h-1)l^2 + 3(h+1)l + 6(h+1)), \\ 6(C + A + B + D - J - Z) &= h(h-1)(3l^2 - h - 1) \ge 0.\end{aligned}$$

The cube fact now completes the proof. □

Formal correspondence: `w23_col_shadows; col_nat_population; col_colored_shadow_population`.

Theorem 13.2. *Every depth-four semigroup with $K(K-1) \le 3l(l+1)$ satisfies Wilf's inequality.*

Proof. Use the graded counts with k_2 now denoting the total number of level-two primitives, and $x = x_h + x_o$. The mixed inequalities in Lemma 11.1 hold without splitting k_2 into strip and ordinary parts. Denote their slacks by

$$B = k_2K - x - y,\ C = k_2(k_2+1) - 2(f + d_{CC}),\ D = Kz - r,\ E = k_2z - s,$$

$$G = k_2b - H_0, \quad M = k_2(b + w_\ell) + (K + k_2 + 2)(x + f) - 3(H_0 + H_1 + j + g), \quad S = \rho - f - H_1 - j - g - s.$$

Expansion of the exact counts gives

$$\begin{aligned}W_0 = F + (Kk_2 - x) + k_2(w - w_\ell + v) + z(1 + z + 2b + w + v + x + f) \\ + 4B + 2C + 4D + 3E + G + M + S.\end{aligned}$$

All terms are nonnegative: $Kk_2 - x \ge 0$ follows from $B \ge 0$ and $y \ge 0$, and Lemma 13.1 gives $F \ge 0$. Hence $W_0 \ge 0$, which implies $W \ge 0$. □

Formal correspondence: `col_general_profile_comparison; w23_q4_gen_lowpop`.

14. An exhaustive depth-four conclusion

We can now simplify the formal source's dispatcher. The anchor argument and the strip extension used only the quartic estimate $d_4 \ge h_4$ and the existence of a high offset. The two other residual hypotheses in (15), namely $K \ge 4$ and the condition on small offsets, never enter those proofs.

Suppose $3l(l+1) < K(K-1)$. If $K \ge 3$, then $h \ge 3$: otherwise $l \ge 1$ and $K = l + h \le l + 2$, giving

$$K(K-1) \le (l+2)(l+1) \le 3l(l+1),$$

a contradiction. Thus the quartic proof applies. If $K < 3$, the strict inequality forces $K = 2, l = 0, h = 2$. Every offset is high; an unwrapped sum has at most one high factor, so $h_3 = h_4 = 0$ and $d_4 \ge h_4$ holds directly. A high anchor still exists. Therefore *every* full-offset core with strict population satisfies

$$W_0(T) + p_T \ge 0, \qquad 13W_0(T) + 23p_T \ge 0. \tag{52}$$

This is a consequence of the proved inequalities, with no need to invoke the source's separate pruning or small-generator branches.

Theorem 14.1. *If $3m < c \le 4m$, then $c \le e\ell$.*

Proof. The weak population case is Theorem 13.2. In the strict case form the core T from Section 12. It has exactly the same first-level primitives and the same low/high split, so (52) applies.

If $W_0(S) \geq 0$, the result is immediate. Otherwise, if $S = T$, then $K \geq 2$ and the exact left count gives $\ell_S \geq 4 + 3K \geq 10$. Thus $W = W_0 + p(\ell - 4) \geq W_0 + p \geq 0$. If $S \neq T$, some left primitive lies outside A, so $k + z \geq 1$. Bound (41) gives $K + 1 \geq 6$ and hence $\ell_S \geq 4 + 3K \geq 19$. The transfer inequality and (44) give

$$13W_0(S) + 23p_S \geq 13W_0(T) + 23p_T \geq 0.$$

Finally $13W = 13W_0 + 13p_S(\ell_S - 4) \geq 13W_0 + 26p_S \geq 0$. □

Combining Theorem 14.1 with Theorem 2.2 and the established depth-three theorem of Eliahou [2] proves (1). The submitted Lean source also develops the depth-three theorem internally. Here that established result is used with its exact hypothesis $c \leq 3m$; the new depth-four argument has been supplied in full above.

15. Source correspondence and review boundaries

The primary source is the submitted solution [1]. The line numbers below refer to the local 22,763-line copy used for this exposition; declaration names are the stable identifiers if formatting changes.

Mathematical step	Formal declaration	Prose status
Primitive generation	`w22_sum_of_primitives`	Proved here
Depth at most two	`w22_kaplan`	Proved here
Finite level model	`wm_B_spec`, `wm_B_of_sum`	Proved here
Model-semigroup link	`wm_of_core` (17402)	Invariant argument given
Divisor closure	`wm_M_parent`	Proved here
Cubic deficit	`wm_d3pp_nonneg`	Proved here
Quartic inequality	`wm_quartic` (18606)	Proved here
Accounting identity	`wm_ID2` (18059)	Proved here
Anchor balance	`an_bal_left`, `an_colrank`	Proved here
Anchor injections	`an_payment` (20510)	Proved here
Strip-free core	`an_anchor_core` (20591)	Proved here
Strip extension	`st_strip_lemma` (22447), `st_A29A` (22478)	Proved here
Generalized core transfer	`cr_thmA` (16826)	Proved here
Residual assembly	`cr_core_reduction` (16922)	Proved here
All depth-four cases	`wilf_q_four` (22682)	Simplified proof here
Universal target	`target` (22753)	Formal statement mapped

This manuscript does not treat comments in the Lean source as proof obligations or proof certificates. Several comments still describe old nodes as open; the corresponding declarations must be assessed by their actual proof terms and dependencies. Conversely, an accepted formal declaration does not automatically make an independently written explanatory proof correct. The reconstructed arguments require mathematical review of their translation, especially boundary inequalities, the use of integer intervals, and multiplicities in sorted cubic representations.

The prose reconstruction has been reviewed by a separate AI agent, including the two anchor injections, the strip-count cancellation, the fibre bridge, and the simplified exhaustive reduction.

The finite scalar minima and polynomial identities were also checked by exact rational and symbolic arithmetic. These checks support the exposition but do not amount to a new kernel verification of the Lean submission. The source's established depth-three branch is cited here through Eliahou's mathematical paper; the depth-four proof is developed explicitly in this manuscript.

References

[1] Elias, submitted formal solution for Wilf's inequality with $c \leq 4m$, Conjectures submission `fd12daa7-2b0e-460f-a805-f9ba6d9fede4`. https://conjectures.io/results/fd12daa7-2b0e-460f-a805-f9ba6d9fede4/solution.

[2] Shalom Eliahou, *Wilf's conjecture and Macaulay's theorem*, arXiv:1703.01761 (2017). https://arxiv.org/abs/1703.01761.